Pergamon

Topology Vol. 36, No. 5, pp. 1123–1153, 1997

0040-9383/97 $17.00 + 0.00

PII: S0040-9383(96)00039-0

CANONICAL REPRESENTATIVES FOR PATTERNS OF TREE MAPS

LLUÍS ALSEDÀ†, JOHN GUASCHI, JÉRÔME LOS, FRANCESC MAÑOSAS and PERE MUMBRÚ

(*Received* 21 *July* 1995)

We define a notion of *pattern* for finite invariant sets of continuous maps of finite trees. A pattern is essentially a homotopy class relative to the finite invariant set. Given such a pattern, we prove that the class of tree maps which exhibit this pattern admits a canonical representative, that is a tree and a continuous map on this tree, which satisfies several minimality properties. For instance, it minimizes topological entropy in its class and its dynamics are minimal in a sense to be defined. We also give a formula to compute the minimal topological entropy directly from the combinatorial data of the pattern. Finally we prove a characterization theorem for zero entropy patterns.

1. INTRODUCTION

One-dimensional dynamics have been studied intensively during the last three decades. The characterization by Sharkovskii [28] of the possible sets of periods of continuous maps of an interval I has generated many questions and results (for instance see [4] or [1] for a review). If $f: I \to I$ is such a map and if A is a finite invariant set of f then intrinsic information can be obtained by considering the "*pattern*" of A which is characterized essentially by the permutation π_A induced by $f|_A$ (see [4, 27] for a precise definition). To each pattern π_A we may associate a (non-unique) interval map f_π which admits a finite invariant set B, such that the permutation induced by $f_\pi|_B$ is π_A and f_π is monotone between consecutive points of B. Such a map is called a *canonical representative of* π_A, or a "connect-the-dots" map. It has the following important properties:

(A) f_π minimizes topological entropy within the class of interval maps admitting an invariant set whose pattern is π_A.
(B) f_π admits a Markov partition which gives a good "coding" to describe the dynamics of the map f_π. The topological entropy of f_π may be calculated from this partition.
(C) f_π is essentially unique.
(D) The pattern of A forces a pattern ρ if and only if f_π has an invariant set pattern is ρ. We recall the definition that a pattern A *forces* a pattern B if and only if each map exhibiting the pattern A also exhibits the pattern B (see [1, 27]). In this sense, the dynamics of f_π are minimal within the class of maps admitting an invariant set whose pattern is π_A.

One may make analogies with the study of surface homeomorphisms. In this case, canonical representatives are given by the Nielsen–Thurston Theorem [15, 30]. In this

†Corresponding author.

context the "pattern" or *braid type* $\mathrm{bt}(f, A)$ of a periodic orbit A of a surface homeomorphism $f: M \to M$ is characterized by the isotopy class (up to conjugacy) of $f|_{M\setminus A}$ [10, 12, 25]. The permutation group arising in the interval case is now replaced by the mapping class group of $M\setminus A$. Analogues of the properties (A)–(D) above are satisfied for each canonical homeomorphism (see [15] for the analogue of properties (A)–(C) and [3, 19] for the analogue of (D)). Remarkably, each canonical homeomorphism has an underlying one-dimensional structure, and its dynamics may be calculated from a particular class of continuous graph maps (the fundamental group of the graph is identified with that of the surface) [5, 24]. In view of this fact it is natural to consider the class of continuous graph maps. The aim of this paper is to carry out the following programme for the simplest class of graphs, namely that of *trees*. We first give a natural definition of the pattern of a finite invariant set of a continuous map of a tree into itself. We then prove the existence of *canonical representatives* satisfying properties analogous to properties (A)–(C) above.

A "connect-the-dots" interval map is characterized by the fact that it is monotone between consecutive points of the given invariant set. For trees, this suggests that we also need a notion of "consecutive" points. In other words, we should fix the relative positions of the points of the invariant set rather than the tree itself. In fact, even if the relative positions of its points are fixed, the invariant set can live in (infinitely) many different trees. We now extend informally these notions to continuous map of a tree into itself. In Section 2 we will give precise versions of all these concepts.

Let T be a tree, and let $A \subset T$ be a finite set. The pair (T, A) will be called a *pointed tree*. Let $\theta: A \to A$ be a map. Consider the homotopy class of (T, A) relative to A. Both this class and the map θ are well defined up to conjugacies. We denote their conjugacy classes by $\mathscr{T} = [T, A]$ and $\Theta = [\theta]$ respectively. The *pattern* of A will be the pair $(\mathscr{T}, \Theta)$. For practical reasons, the formal definition of a pattern will be combinatorial. The relative positions of the points of A is taken into account via the notion of homotopy class of pointed trees.

Let (T, A) be a pointed tree. Each maximal subset of A contained in the closure of a connected component of $T\setminus A$ will be called a *discrete component*. We formalize the intuitive concept of "consecutive" points as follows. Any binary subset of a discrete component will be called a *basic path*. Let $f: T \to T$ be a tree map such that $f(A) \subset A$. For each basic path π, the unique locally injective path joining the points of π is denoted by $\langle\pi\rangle$. The fact that f is an analogue of the "connect-the-dots" map with respect to A can be formulated naturally as follows: for any basic path π, $f(\langle\pi\rangle)$ is an interval, and the restriction of f to $\langle\pi\rangle$ is monotone. Such a map will be called an *A-monotone* map.

For each interval pattern there exists an A-monotone interval map exhibiting this pattern (the "connect-the-dots" map). However, if $f: T \to T$ is a continuous tree map exhibiting a pattern over $A \subset T$ then there does not always exist a continuous A-monotone tree map of the *same* pointed tree (T, A). Below we give an example of such a situation.

Example 1.1. Consider the map $f: T \to T$ where T is the tree shown in Fig. 1 and $f(x_i) = x_{i+1}$ for $i = 1, \dots, 5$ and $f(x_6) = x_1$. Suppose that the map f is A-monotone with respect to the set $A = \{x_1, x_2, \dots, x_6\}$. In particular, we have that

$$f(\langle x_1, x_2\rangle) = \langle x_2, x_3\rangle$$

$$f(\langle x_3, x_5\rangle) = \langle x_4, x_6\rangle.$$

Hence, $f(v) \in \langle x_2, x_3\rangle$ and $f(v) \in \langle x_4, x_6\rangle$. This gives a contradiction.

In view of the above example, a natural question is: given a pattern, does there always exist a tree in which the pattern admits an A-monotone map? The main aim of

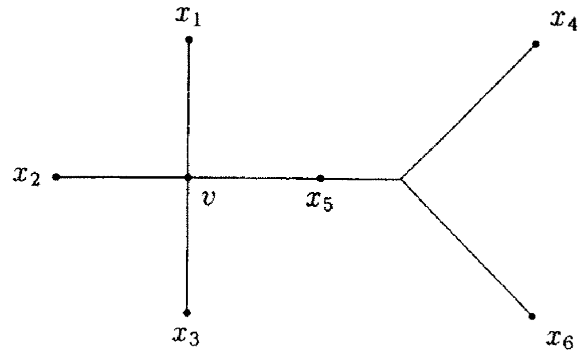

Fig. 1. The tree T of Example 1.1.

this paper is to answer this question in the affirmative (see Theorem A). Indeed we prove the following.

Given a pattern $(\mathscr{T}, \Theta)$, *there exists a tree* T, *a finite set* $A \subset T$ *and an* A*-monotone continuous map* $f: T \to T$ *such that* $(\mathscr{T}, \Theta) = ([T, A], [f|_A])$. *Moreover, the topological entropy* $h(f)$ *of* f *satisfies*

$$h(f) = \inf\{h(g): g \text{ exhibits } (\mathscr{T}, \Theta)\} =: h(\mathscr{T}, \Theta).$$

We also prove that the topological entropy of such a map can be computed directly from a pattern invariant called the *path transition matrix*. This matrix, denoted by $M_P(\mathscr{T}, \Theta)$, depends only on the pattern and not on any of its realizations (see Section 3). We then show that

$$h(\mathscr{T}, \Theta) = \log \max\{\rho(M_P(\mathscr{T}, \Theta)), 1\}$$

where $\rho(M)$ denotes the spectral radius of the matrix M.

So there exist A-monotone maps satisfying Property (A). Analogues of Properties (B) and (C) also hold (see Theorem B). These results are really analogues of Young's Theorem [31] for interval maps and of Thurston's Theorem for surface homeomorphisms. There is nevertheless a noticeable difference between our results and the two-dimensional ones mentioned previously, namely that the minimal topological entropy is computed from the path transition matrix rather than from a Markov transition matrix (for interval maps, these two matrices are the same). The path transition matrix is defined by the pattern, i.e. for a whole class of maps, in contrast to the Markov transition matrix which is only defined for certain specific maps. In fact the path transition matrix carries essentially the same information as the Markov transition matrix of the canonical model. For example, it contains all of the information concerning the periodic orbits of the canonical representative (see Theorems C and D in the following section). As a corollary of the main result, we obtain a complete characterization of zero entropy patterns (Theorem E) which is very similar to that for surface homeomorphisms [16, 18, 22] and interval maps [6, 11, 26].

The proof of the main result is constructive, this means that we give a finite process to build an A-monotone tree map which exhibits the given pattern. Furthermore, the A-monotonicity property implies the existence of a natural Markov partition for this map.

In this setting several problems remain open, among them:

(1) What is the appropriate definition of forcing for patterns of tree maps in order that an analogue of Property (D) holds?

(2) Is it possible to obtain models satisfying Properties (A), (B) and (C) keeping the tree fixed? In general, the entropy of such models will be larger than that of the A-monotone ones.

2. DEFINITIONS AND STATEMENT OF THE RESULTS

In this section we will introduce the definitions and state the results mentioned in the introduction in detail.

By an *interval* we mean the closed interval $[0, 1]$ and any space homeomorphic to it. A (finite) *tree* is a uniquely arcwise-connected space that is a point or a union of a finite number of intervals. Let T be a tree. Given a point $x \in T$ we define the *valence* of x, denoted by $\mathrm{Val}_T(x)$, to be the number of connected components of $T \setminus \{x\}$. By convention, if T consists of a single point x then we set $\mathrm{Val}_T(x) = 0$. An *interior* point of T is a point whose valence is at least two. Each point whose valence is at most one will be called an *endpoint* of T, and the set of endpoints will be denoted by $\mathrm{En}(T)$. Each point whose valence is different from two will be called a *vertex* of T, and the set of vertices of T will be denoted by $V(T)$. The closure of each connected component of $T \setminus V(T)$ will be called an *edge* of T. Given $A \subset T$ we will define the *convex hull* of A, denoted by $\langle A \rangle_T$ or by $\langle A \rangle$, to be the smallest closed connected set containing A. When $A = \{x, y\}$ we will write simply $\langle x, y \rangle$ to denote $\langle A \rangle$. Also $\mathrm{Int}(B)$ and $\mathrm{Cl}(B)$ will denote the interior and the closure respectively of a set B. If A is finite then we denote its cardinality by $|A|$. For $n \geqslant 3$, an *n-star* will be a tree with n endpoints and a unique vertex of valence n. By convention, a tree consisting of a single point will be called a 1-*star*, and a tree consisting of a single edge will be called a 2-*star*.

Let T be a tree, and let A be a finite subset of T. The pair (T, A) will be called a *pointed tree*. A nonempty set $Q \subset A$ is said to be a *discrete component* of (T, A) if either $|Q| > 1$ and there exists a connected component C of $T \setminus A$ such that $Q = A \cap \mathrm{Cl}(C)$, or $|Q| = 1$ and $Q = A$. If Q and Q' are two distinct discrete components of (T, A) then $Q \cap Q'$ is either empty or contains a unique element. Any (unordered) binary subset of a discrete component will be called a *basic path* of (T, A).

We now introduce an equivalence relation on pointed tress. We say that two pointed trees (T, A) and (T', A') are *equivalent*, written $(T, A) \sim (T', A')$, if there exists a bijection $\phi: A \to A'$ which preserves discrete components. Thus Q is a discrete component of (T, A) if and only if $\phi(Q)$ is a discrete component of (T', A'). The equivalence class of a pointed tree (T, A) will be denoted by $[T, A]$. With this definition, the topology of T and T' may be different. We will see in Lemma 6.1 that the fact that ϕ preserves discrete components is equivalent to:

(i) π is a basic path of (T, A) if and only if $\phi(\pi)$ is a basic path of (T', A'), and
(ii) for each $a, b, c \in A$ one has $a \in \langle b, c \rangle_T$ if and only if $\phi(a) \in \langle \phi(b), \phi(c) \rangle_{T'}$.

A tree may be thought as a topological space, but also combinatorially as a set of vertices with an adjacency relation (edges). We whall consider trees in both ways. In this context, if two trees S and T are homeomorphic (as topological spaces) then they have the same combinatorial structure.

Remark 2.1. Let (T, A) and (T', A') be equivalent pointed trees. Let Q and Q' be discrete components of (T, A) such that $Q \cap Q' = \{z\}$. If $\phi: A \to A'$ is a bijection which preserves discrete components then $\phi(Q)$ and $\phi(Q')$ are discrete components of (T', A'), and $\phi(Q) \cap \phi(Q') = \{\phi(z)\}$. Consequently ϕ associates the discrete components of (T, A) with the discrete components of (T', A') and preserves their relative positions. More precisely, let Q be a discrete component of (T, A) and let $Z \subset Q$ be such that each element z of Z belongs to a discrete component Q_z of (T, A) different from Q. Then $\phi(Q)$ is a discrete component of (T', A') such that $|\phi(Q)| = |Q|$ and $\phi(Z)$ is the subset of $\phi(Q)$ each of whose elements belongs to a discrete component $\phi(Q_z)$ of (T', A') different from $\phi(Q)$, and $\phi(Q) \cap \phi(Q_z) = \{\phi(z)\}$ for each $z \in Z$.

As we have seen, a class of pointed trees is determined solely by the discrete components and their relative positions. Notice also that a connected component of $T \setminus A$ with only one point of its closure in A can be retracted to this point. In other words, we can always assume that $\mathrm{En}(T) \subset A$.

Let (T, A) and (T', A') be equivalent pointed trees, and let $\theta: A \to A$ and $\theta': A' \to A'$ be maps. We will say that θ and θ' are *equivalent*, written $\theta \approx \theta'$, if and only if $\theta' = \phi \circ \theta \circ \phi^{-1}$ for a bijection $\phi: A \to A'$ which preserves discrete components. The equivalence class of θ by the relation $\approx$ will be denoted by $[\theta]$. If $\mathscr{T} = [T, A]$ is an equivalence class of pointed trees and if $\Theta = [\theta]$ is an equivalence class of maps then the pair $(\mathscr{T}, \Theta)$ will be called a *pattern.*

In what follows, any continuous map from a tree to itself will be called a *tree map*. Let T be a tree, and let $f: T \to T$ be a tree map. We say that f *exhibits* the pattern $(\mathscr{T}, \Theta)$ if there exists a finite f-invariant subset A of T such that $\mathscr{T} = [T, A]$ and $\Theta = [f|_A]$. We will also say that f *exhibits* $(\mathscr{T}, \Theta)$ *over* A.

To each pattern $(\mathscr{T}, \Theta) = ([T, A], [\theta])$, we will associate a *path transition matrix* as follows. Let $\{\pi_1, \pi_2, \dots, \pi_k\}$ be the set of basic paths of (T, A). The $k \times k$ matrix $M_{\mathrm{P}}(\mathscr{T}, \Theta) = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } \pi_j \subset \langle \theta(\pi_i) \rangle_T \\ 0 & \text{otherwise} \end{cases}$$

will be called the *path transition matrix* of $(\mathscr{T}, \Theta)$. From above it follows that this definition is independent of the particular choice of T, A and θ. Thus the path transition matrix is a well-defined pattern invariant.

Let S, T be trees, and let $f: T \to S$ be a continuous map. If $a, b \in T$ then we say that $f|_{\langle a, b \rangle}$ is *monotone* if $f(\langle a, b \rangle)$ is homeomorphic to an interval (perhaps degenerate) and $f|_{\langle a, b \rangle}$ is monotone as an interval map. Now let $A \subset T$ be a finite set such that $A \supset \mathrm{En}(T)$. We say that f is *A-monotone* if $f(\langle \pi \rangle_T) = \langle f(\pi) \rangle_S$ and $f|_{\langle \pi \rangle_T}$ is monotone for each basic path π of (T, A). In Proposition 4.2 we shall see that the fact that a tree map f exhibiting the pattern $([T, A], \Theta)$ is A-monotone implies that the image of each vertex is uniquely determined and is either a vertex or belongs to A.

The *topological entropy* of a tree map f will be denoted by $h(f)$ (see e.g. [13] for a definition). We define the *entropy* of a pattern $(\mathscr{T}, \Theta)$ by

$$h(\mathscr{T}, \Theta) = \inf\{h(f): f \text{ is a tree map which exhibits } (\mathscr{T}, \Theta)\}.$$

Let M be a square matrix. We shall denote by $\rho(M)$ the spectral radius of M. The main result of this paper is the following.

THEOREM A. *Let $(\mathscr{T}, \Theta)$ be a pattern. Then the following statements hold.*

(a) $h(\mathscr{T}, \Theta) = \log \max\{\rho(M_{\mathrm{P}}(\mathscr{T}, \Theta)), 1\}$.
(b) $h(f) = h(\mathscr{T}, \Theta)$ *for each tree map f which exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone.*
(c) *There exist a pointed tree (T, A) and an A-monotone tree map $f: T \to T$ such that it exhibits $(\mathscr{T}, \Theta)$ over A.*

It follows that the entropy of a pattern can be computed *directly* from the pattern by means of the associated path transition matrix. This is in contrast to the situation for surface homeomorphisms, where the minimal entropy is also computable, but requires a non-trivial algorithm.

The proof of Theorem A will follow immediately (see Section 5) from three results (Theorems 3.1, 4.1 and 5.1) which correspond essentially to the three statements of the theorem. Theorem A generalizes the known results for interval maps (see [1], for example) and the results of Li and Ye (see [21]) concerning continuous maps of 3-stars. These are the only cases for which the A-monotone models are obtained without modifying the tree.

From Theorem A we see that there exist A-monotone models for each pattern. These models are unique to the following equivalence relation. Let $(\mathcal{T}, \Theta)$ be a pattern, and let $f: T \to T$ be an A-monotone tree map which exhibits the pattern $(\mathcal{T}, \Theta)$ over A. We will say that $v_1, v_2 \in V(T)\backslash A$ are *f-identifiable* if either:

(i) $\langle f^i(v_1), f^i(v_2)\rangle \cap A = \emptyset$ for all $i \geqslant 0$, or
(ii) if $\langle f^n(v_1), f^n(v_2)\rangle \cap A \neq \emptyset$ for some $n \geqslant 0$ then $f^n(v_1) = f^n(v_2)$.

Let $(\mathcal{T}, \Theta)$ be a pattern, let (T, A) be a pointed tree, and let $f: T \to T$ be a tree map. The triplet (T, A, f) will be called a *canonical model* of $(\mathcal{T}, \Theta)$ if the following conditions hold:

(i) f exhibits the pattern $(\mathcal{T}, \Theta)$ over A and is A-monotone.
(ii) if $v, w \in V(T)\backslash A$ are f-identifiable then $v = w$.

Let (T, A) be a pointed tree, and let $f: T \to T$ be an A-monotone tree map such that $f(A \cup V(T)) \subset A \cup V(T)$. Observe that f is $A \cup V(T)$-monotone. Then the map admits a Markov partition given by the closures of the connected components of $T\backslash(A \cup V(T))$ (see for instance [8]), and we denote the corresponding Markov transition matrix by $M_{\mathrm{M}}((T, A \cup V(T)), f)$. In Section 6 we will see that a canonical model is obtained from any A-monotone model by collapsing edges which join identifiable vertices to a point. More explicitly, we prove the following theorem.

THEOREM B. *Let $(\mathcal{T}, \Theta)$ be a pattern. Then*

(a) *there exists a canonical model of $(\mathcal{T}, \Theta)$.*
(b) *any canonical model (T, A, f) of $(\mathcal{T}, \Theta)$ admits a Markov partition given by the closures of the connected components of $T\backslash(A \cup V(T))$.*
(c) *given two canonical models (T, A, f) and (T', A', f') of $(\mathcal{T}, \Theta)$ there exists a homeomorphism $h: T \to T'$ such that $h(A) = A'$, and $f' \circ h|_A = h \circ f|_A$. Moreover, $M_{\mathrm{M}}((T, A \cup V(T)), f) = M_{\mathrm{M}}((T', A' \cup V(T')), f')$ (up to relabelling).*

The canonical models for interval maps and surface homeomorphisms have "minimal dynamics" (see for instance Theorem 2.6.4 of [1]; see also [3]). In some sense this also happens for A-monotone tree maps, as we shall see presently. Let $f: T \to T$ be a tree map, and let $x, y \in T$ be fixed points of f^n for some $n \in \mathbb{N}$. We say that x and y are *f-monotone equivalent* if either $x = y$ or $f^n|_{\langle x, y\rangle}$ is monotone. The notion of monotone equivalence is in the same spirit as the notion of Nielsen equivalence for surface homeomorphisms [20]. We note that $f^m(x)$ and $f^m(y)$ are also f-monotone equivalent for all $m \geqslant 0$. When $\langle x, y\rangle \cap V(T) = \emptyset$ it is easy to see (as in the interval case) that the periods of x and y are either equal, or one is twice that of the other. However, for each $k \in \mathbb{N}$ one may construct an example of a tree map $f_k: T \to T$ such that there exist f_k-monotone equivalent points $x \in V(T)$ and $y \in T$ for which the period of y is k times that of x (for example, rigid rotation of a k-star).

Let $(\mathscr{T}, \Theta)$ be a pattern, let $f: T \to T$ be a tree map which exhibits the pattern over A, and let x be a periodic point of f. We will say that x is *significant* if it is not f-monotone equivalent to any element of $V(T) \cup A$ and its period is minimal within its f-monotone equivalence class. The $([T, A], [\theta])$-*path graph* is the oriented graph whose vertices are in one-to-one correspondence with the basic paths of (T, A), and there is an oriented edge (or arrow) from the vertex i to the vertex j if and only if the corresponding basic paths satisfy $\pi_j \subset \langle \theta(\pi_i) \rangle$. In other words, $M_P([T, A], [\theta])$ is the transition matrix of the $([T, A], [\theta])$-path graph. For a (combinatorial) graph we define a *path of length* m to be a sequence of vertices $(V_0, V_1, \dots, V_m)$ such that there is an arrow from V_j to V_{j+1} for each $j = 0, 1, \dots, m-1$. Usually such a path will be written as $V_0 \to V_1 \to \cdots \to V_m$. A *loop* is a path of the form $V_0 \to V_1 \to \cdots \to V_{m-1} \to V_0$.

Let $\alpha = V_0 \to V_1 \to \cdots \to V_{m-1} \to V_0$ and $\beta = U_0 \to U_1 \to \cdots \to U_{n-1} \to U_0$ be two loops such that $V_0 = U_0$. Then we denote the loop $V_0 \to V_1 \to \cdots \to V_{m-1} \to V_0 \to U_1 \to \cdots \to U_{n-1} \to V_0$ by $\alpha\beta$. A loop will be called *non-repetitive* if it cannot be written as α^n for some loop α with $n > 1$.

Let $(\mathscr{T}, \Theta)$ be a pattern, and let $\alpha = \pi_0 \to \cdots \to \pi_{n-1} \to \pi_0$ be a loop of length n of the $(\mathscr{T}, \Theta)$-path graph. Let $f: T \to T$ be a tree map which exhibits the pattern $(\mathscr{T}, \Theta)$. We will say that $x \in T$ and α are *associated* if $f^i(x) \in \langle \pi_{i \pmod n} \rangle$ for each $i \geqslant 0$.

Now we are ready to state the two results which describe the sense in which the A-monotone maps have minimal dynamics. They will be proved in Section 7.

THEOREM C. *Let $(\mathscr{T}, \Theta)$ be a pattern and let $f: T \to T$ be an A-monotone tree map exhibiting the pattern $(\mathscr{T}, \Theta)$ over A. Then the following statements hold.*

(a) *For each significant point x of f of period n there exists a unique non-repetitive loop α of length n of the $(\mathscr{T}, \Theta)$-path graph such that x and α are associated.*
(b) *Each non-repetitive loop α of length n of the $(\mathscr{T}, \Theta)$-path graph is associated either to a significant point of f of period n or to a periodic point which is f-monotone equivalent to a point of $V(T) \cup A$ and whose period is a divisor of n. In both cases, the point associated to α is unique up to f-monotone equivalence.*

THEOREM D. *Let $(\mathscr{T}, \Theta)$ be a pattern, and let f be a tree map exhibiting $(\mathscr{T}, \Theta)$. Let α be a non-repetitive loop of length n of the $(\mathscr{T}, \Theta)$-path graph. Then there exists a fixed point x of f^{2n} such that α and x are associated.*

We will also prove a characterization theorem for zero entropy patterns, i.e. for patterns $(\mathscr{T}, \Theta)$ for which $h(\mathscr{T}, \Theta) = 0$. It turns out that this characterization is very similar to that for surface homeomorphisms (see for instance [16, 18]). Let $(\mathscr{T}, \Theta) = ([T, A], [\theta])$ be a pattern, and let π be a basic path of (T, A). We say that $(\mathscr{T}, \Theta)$ is π-*reducible* (or simply *reducible* if no precision is necessary) if $\theta^n(\pi)$ is contained in a single discrete component for each $n \geqslant 0$. Given such a pattern we define a *reduced* pattern as follows. By Theorem A we can assume that there exists a tree $\tilde{T}$ and a tree map $f: \tilde{T} \to \tilde{T}$ such that f exhibits $(\mathscr{T}, \Theta)$ over A and is A-monotone. Let $C = \bigcup_{n=0}^{\infty} \langle \theta^n(\pi) \rangle_{\tilde{T}}$, and let $C_1, C_2, \dots, C_k$ denote the connected components of C. Since f is A-monotone, the set C is f-invariant. Since the f-image of a connected set is connected, for each $i \in \{1, 2, \dots, k\}$, there exists j_i such that $f(C_i) \subset C_{j_i}$. Then take T' to be the tree obtained from $\tilde{T}$ by collapsing each C_i to a point c_i, and let $\phi: \tilde{T} \to T'$ denote the canonical projection. Then ϕ is continuous, it is bijective on $\tilde{T} \setminus C$, and $\phi(C_i) = c_i$ for each $i = 1, 2, \dots, k$. Set $A' = \phi(A)$. Define $\theta': A' \to A'$ by $\theta' \circ \phi|_A = \phi \circ \theta$. So θ' is well defined because $f|_A = \theta$. The pattern $([T', A'], [\theta'])$ depends on the chosen A-monotone model, and more precisely on the tree $\tilde{T}$. It will be called the $(\pi, \tilde{T})$-*reduced* (or simply a *reduced* pattern) of $(\mathcal{T}, \Theta)$, and the process of obtaining $([T', A'], [\theta'])$ from $(\mathcal{T}, \Theta)$ will be called a *reduction*. The pattern $([T, A], [\theta])$ will be called *trivial* if A consists of a single point and *strongly reducible* if there exists a finite sequence of reductions to a trivial pattern.

The notion of a strongly reducible pattern depends apparently on the chosen sequence of basic paths and A-monotone models. From the next theorem, which characterizes the zero entropy patterns, it follows that this notion is well defined.

THEOREM E. *A pattern has zero entropy if and only if it is strongly reducible.*

The characterization of zero entropy patterns (not just those associated to periodic orbits or permutations; see for instance [7]) given by Theorem E was unknown even in the case of interval and star maps.

As we indicated in the Introduction, one may ask similar questions in the case where the tree is fixed. There exist two types of trees (intervals and 3-stars) which remain unchanged under modifications which preserve the pattern. In general there are many different trees T admitting a given pattern $(\mathcal{T}, \Theta) = ([T, A], [\theta])$. If $\langle A \rangle_T$ is an interval (respectively a 3-star) then $\langle A' \rangle_{T'}$ is an interval (respectively a 3-star) for all T', A' and θ' such that $(\mathcal{T}, \Theta) = ([T', A'], [\theta'])$. Thus Theorem E also characterizes the zero entropy patterns when the tree is an interval or a 3-star; in particular, it characterizes zero entropy periodic orbits in these cases. Other characterizations of zero entropy periodic orbits for interval and star maps may be found in [1, 2, 9, 23, 27] for instance. One may also obtain a new proof of the following well-known result about the periods of zero entropy periodic orbits using Theorem E.

COROLLARY F. *Let* $([T, A], [\theta])$ *be a pattern such that* $\langle A \rangle_T$ *is an s-star for some* $2 \leqslant s \leqslant 3$ *and* A *is a periodic orbit. If* $h([T, A], [\theta]) = 0$ *then* $|A| = r2^k$ *for non-negative integers* k *and* $1 \leqslant r \leqslant s$.

Theorem E and Corollary F will be proved in Section 8.

3. LOWER BOUNDS OF ENTROPY FOR TREE MAPS EXHIBITING A PATTERN

In this section we prove the following result which will be used in the proof of Theorem A.

THEOREM 3.1. *Let* f *be a tree map that exhibits the pattern* $(\mathcal{T}, \Theta)$. *Then* $h(f) \geqslant \log \max\{\rho(M_P(\mathcal{T}, \Theta)), 1\}$.

We start by stating and proving a technical lemma.

LEMMA 3.2. *For* $j = 1, \ldots, s$ *let* $\pi_0 \to \pi_1^j \to \cdots \to \pi_n^j$ *be s different paths of length n in the* $(\mathcal{T}, \Theta)$*-path graph, and let* $f\colon T \to T$ *be a tree map which exhibits the pattern* $(\mathcal{T}, \Theta)$ *over* A. *Then there exists subsets* $J^1, \ldots, J^s$ *of* $\langle \pi_0 \rangle$ *which are finite unions of closed intervals such that for each* $j = 1, \ldots, s$ *we have that* $J^j \subset \langle \pi_0 \rangle$, *and* $f^i(J^j) \subset \langle \pi_i^j \rangle$ *for* $i = 1, \ldots, n-1$, *and* $f^n(J^j) = \langle \pi_n^j \rangle$. *Furthermore,* $\mathrm{Int}(\langle J^k \rangle) \cap \mathrm{Int}(\langle J^j \rangle) = 0$ *for any* $j, k \in \{1, \ldots, s\}$ *with* $j \neq k$.

Proof. To prove the lemma we use induction on n. Let $n = 1$ and set $\langle \pi_0 \rangle = \langle a_0, b_0 \rangle$ and $\langle \pi_1^j \rangle = \langle a_j, b_j \rangle$ for $j = 1, \ldots, s$. Clearly, for any $j = 1, \ldots, s$ we have that $\langle a_j, b_j \rangle \subset \langle f(a_0), f(b_0) \rangle$. We consider the interval $\langle f(a_0), f(b_0) \rangle$ to be endowed with the orientation such that $f(a_0) < f(b_0)$. Without loss of generality, we may assume that $f(a_0) \leqslant a_1 < b_1 \leqslant \cdots \leqslant a_s < b_s \leqslant f(b_0)$. We also consider $\langle a_0, b_0 \rangle$ to be oriented so that $a_0 < b_0$. For $j = 1, \ldots, s$ set

$$a_j' = \inf\{x \in \langle a_0, b_0 \rangle;\ f(x) = a_j\}$$

and

$$b_j' = \inf\{x \in \langle a_0, b_0 \rangle;\ f(x) = b_j\}.$$

Then we see that $a_0 \leqslant a_1' < b_1' \leqslant \cdots \leqslant a_s' < b_s' \leqslant b_0$. Let $j \in \{1, \ldots, s\}$, and suppose that there are $k \geqslant 0$ elements $c_1, \ldots, c_k$ of $V(T) \cap \mathrm{Int}(\langle a_j, b_j \rangle)$ such that $a_j < c_1 < \cdots < c_k < b_j$. Set $c_0 = a_j$ and $c_{k+1} = b_j$. For each $i = 0, 1, \ldots, k$ define

$$\alpha_i = \sup\{x \in [\beta_{i-1}, b_j']\colon f(x) = c_i\}$$

$$\beta_i = \inf\{x \in [\alpha_i, b_j']\colon f(x) = c_{i+1}\}$$

where $\beta_{-1} = a_j'$. Then $f([\alpha_i, \beta_i]) = [c_i, c_{i+1}]$, and thus $J^j = \bigcup_{i=0}^{k} [\alpha_i, \beta_i]$ satisfies $f(J^j) = \langle a_j, b_j \rangle$. Since $\mathrm{Int}(\langle a_j', b_j' \rangle) \cap \mathrm{Int}(\langle a_l', b_l' \rangle) = \emptyset$ for all $j, l \in \{1, \ldots, s\}$ with $j \neq l$, it is also clear that $\mathrm{Int}(J^j) \cap \mathrm{Int}(J^l) = \emptyset$.

Now suppose that the lemma holds for any $i < n$. For each $j \in \{1, \ldots, s\}$, let $L_j = \pi_0 \to \pi_1^j \to \cdots \to \pi_{n-1}^j$. Since some of the L_j may be equal, let k_j be the number of $l \in \{1, \ldots, s\}$ such that $L_j = L_l$. By relabelling the L_j if necessary, we may choose $r \leqslant s$ so that the paths $L_1, \ldots, L_r$ are different and such that $s = k_1 + \cdots + k_r$. By the induction hypothesis, there exist r subsets $J'^1, \ldots, J'^r$ of $\langle \pi_0 \rangle$ whose interiors are pairwise disjoint, such that for each $m = 1, \ldots, r$ we have

(i) $f^i(J'^m) \subset \langle \pi_i^m \rangle$ for $i = 1, \ldots, n-2$,
(ii) $f^{n-1}(J'^m) = \langle \pi_{n-1}^m \rangle$.

For each such m there are k_m paths $L_m \to \pi_n^j$ of length n, and thus there are k_m different paths $\pi_{n-1}^m \to \pi_n^j$. Using the same construction as in the proof of the case $n = 1$, for each $m \in \{1, \ldots, r\}$ there exist k_m subsets $J^1, \ldots, J^{k_m}$ of J'^m each of which is a finite union of intervals, $f^n(J^i) = \langle \pi_n^i \rangle$ for each $i = 1, \ldots, k_m$, and the interiors of the convex hulls of the J^i are pairwise disjoint. Since $s = k_1 + \cdots + k_r$, this completes the proof. ∎

Remark 3.3. The previous lemma can be strengthened. If instead of taking paths $\pi_0 \to \pi_1^j \to \cdots \to \pi_n^j$ in the $(\mathcal{T}, \Theta)$-path graph, one considers finite sequences of intervals such that the image of each interval covers the following one in the sequence, and that two different sequences satisfy the property that the intervals at the first point where they differ have pairwise disjoint interiors, one may obtain a somewhat stronger statement. The formulation of Lemma 3.2 is however better suited to what follows.

Proof of Theorem 3.1. Let $f\colon T \to T$ be a tree map which exhibits the pattern $(\mathcal{T}, \Theta)$. Let $M_{\mathrm{P}} = M_{\mathrm{P}}(\mathcal{T}, \Theta)$, and let s be the ith entry of the diagonal of M_{P}^n. So there are s different loops of length n in the $(\mathcal{T}, \Theta)$-path graph of f starting at π_i (here we use the notation from the definition of the matrix M_{P}). From Lemma 3.2, there exist s subsets $J_1, \ldots, J_s$ of $\langle \pi_i \rangle$ each of which is a finite union of intervals such that for $j = 1, \ldots, s$, $f^n(J_j) = \langle \pi_i \rangle$ and $\mathrm{Int}\langle J_k \rangle \cap \mathrm{Int}\langle J_j \rangle = \emptyset$ if $j \neq k$. In the terminology of [1], f^n has an s-quasihorseshoe in the interval $\langle \pi_i \rangle$. From Proposition 4.3.2 and Remark 4.3.4 of [1] we obtain that

$h(f) \geqslant (\log s)/n$ and thus $h(f) \geqslant (\log s_n)/n$, where s_n is the largest element in the diagonal of M_P^n. If k is the size of the matrix M_P we have that

$$\log \rho(M_P) = \limsup_{n \to \infty} \frac{\log(\mathrm{Tr}(M^n))}{n} \leqslant \limsup_{n \to \infty} \frac{\log k s_n}{n} = \limsup_{n \to \infty} \frac{\log s_n}{n} \leqslant h(f)$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of the matrix. This ends the proof of the theorem. ■

4. ENTROPY OF A-MONOTONE MAPS

This section is devoted to proving the following theorem.

THEOREM 4.1. *Suppose that $f: T \to T$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. Then $h(f) = \log \max\{\rho(M_P(\mathscr{T}, \Theta)), 1\}$.*

Before starting the proof of Theorem 4.1 we need some technical results. The first one shows that if $f: T \to T$ is A-monotone then the set $A \cup V(T)$ is f-invariant, and hence f is also $A \cup V(T)$-monotone.

PROPOSITION 4.2. *Suppose that $f: T \to T$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. Then for each $z \in A \cup V(T)$ we have that $f(z) \in A \cup V(T)$ and $f(z)$ is uniquely determined.*

Proof. It is enough to consider $z \in V(T) \backslash A$. Since f is A-monotone we have that $\mathrm{En}(T) \subset A$, and hence $\mathrm{Val}(z) \geqslant 3$. Let Q be the discrete component of (T, A) such that $z \in \mathrm{Int}(\langle Q \rangle)$. Since $\mathrm{Val}(z) \geqslant 3$, it follows that there exist $a, b, c \in Q \subset A$ such that $z \in \langle a, b \rangle \cap \langle a, c \rangle \cap \langle b, c \rangle$. We claim that $Z = f(\langle a, b \rangle) \cap f(\langle a, c \rangle) \cap f(\langle b, c \rangle)$ consists of a single point of $A \cup V(T)$. This point must be $f(z)$ and so the statement holds.

Now we prove the claim. Since f is A-monotone, $f(\langle x, y \rangle) = \langle f(x), f(y) \rangle$ for each $x, y \in A$. If $\langle f(a), f(b), f(c) \rangle_T$ is a 3-star then Z consists of a single point of valence at least 3. If not, then there exists $t \in \{a, b, c\}$ such that $f(t) \in f(\langle \{a, b, c\} \backslash \{t\} \rangle)$, and hence $Z = \{f(t)\} \in A$. This ends the proof of the claim. ■

LEMMA 4.3. *Suppose that $f: T \to T$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. Suppose that $\sigma \subset T$ is a binary set contained in the convex hull of a basic path of (T, A) such that $f^n(\sigma)$ reduces to a point for some $n \in \mathbb{N}$ and there exists $u \in \langle \sigma \rangle$ satisfying $f^n(u) \neq f^n(\sigma)$. Then there exists a point $z \in \langle \sigma \rangle$ such that $f^k(z) \in A$ for some $k < n$.*

Proof. Set $\sigma = \{x, y\}$, and let n be the smallest positive integer such that $f^n(x) = f^n(y)$. Since $f^n(u) \neq f^n(\sigma)$, $u \in \langle \sigma \rangle$, σ is contained in the convex hull of a basic path and f is A-monotone, we get that $n > 1$.

Assume that for some $i \in \{1, 2, \ldots, n-1\}$ there does not exist a basic path π' such that $f^i(\sigma) \subset \langle \pi' \rangle$. Then $f^{n-1}(\langle \sigma \rangle) \cap A \neq \emptyset$ and we are done. Otherwise, for each $i \in \{0, 1, 2, \ldots, n-1\}$, $f^i(\sigma)$ is contained in the convex hull of a basic path of (T, A). Since f is A-monotone, we obtain that $f^{n-1}(u) \in f^{n-1}(\langle \sigma \rangle) \subset \langle \pi \rangle$ where π is a basic path of (T, A). Since $f^n(u) \neq f^n(\sigma)$ and f is A-monotone, we have a contradiction. ■

LEMMA 4.4. *Suppose that $f: T \to T$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. Assume that J is an $A \cup V(T)$-basic interval containing $m \geqslant 3$ closed intervals $J_1, J_2, \ldots, J_m$*

whose interiors are pairwise disjoint such that $f^n(J_i) = J$ for some $n \in \mathbb{N}$ and for each $i = 1, 2, \ldots, m$. Then J contains $m - 2$ closed intervals $\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_{m-2}$ whose interiors are pairwise disjoint such that for each $i = 1, 2, \ldots, m-2$, there exists a basic path π_i satisfying $f^n(\tilde{J}_i) = \langle \pi_i \rangle \supset J$ and $f^n(\mathrm{Int}(\tilde{J}_i)) = \mathrm{Int}(\langle \pi_i \rangle)$.

Proof. Without loss of generality, we may suppose that the intervals J_i satisfying the assumptions of the lemma are minimal; in particular, that the endpoints of J_i are mapped to different endpoints of J by f^n for each $i = 1, 2, \ldots, m$. For each $i = 1, 2, \ldots, m$ we take a point $x_i \in \mathrm{Int}(J_i)$ with the property that $f^n(x_1) = f^n(x_2) = \cdots = f^n(x_m) \in \mathrm{Int}(J)$. By relabelling the J_i if necesary, we may suppose that $\langle x_i, x_{i+1} \rangle \cap \{x_i, x_2, \ldots, x_m\} = \{x_i, x_{i+1}\}$ for each $i = 1, 2, \ldots, m-1$. Since each J_i is minimal and satisfies $f^n(J_i) = J$, for each $i \in \{1, 2, \ldots, m-1\}$ there exists a point $u_i \in \langle x_i, x_{i+1} \rangle$ such that $f^n(u_i)$ is an endpoint of J. So for $i, j \in \{1, 2, \ldots, m-1\}$ we have $f^n(u_i) \neq f^n(x_j)$. By Lemma 4.3, for each $i \in \{1, 2, \ldots, m-1\}$, we can define a nonempty set $B_i = \bigcup_{j=1}^{n} f^{-j}(A) \cap \langle x_i, x_{i+1} \rangle$. For each $i = 1, 2, \ldots, m-2$, let $\tilde{J}_i = \langle w_i, v_{i+1} \rangle$ be the minimal interval such that $w_i \in B_i$ and $v_{i+1} \in B_{i+1}$. By definition, if $z \in \mathrm{Int}(\tilde{J}_i)$ then $f^j(z) \notin A$ for each $i = 1, 2, \ldots, m-2$ and $0 \leqslant j \leqslant n$. Therefore $f^n(\tilde{J}_i) = \langle f^n(w_i), f^n(v_{i+1}) \rangle$, and since $f^n(w_i), f^n(v_{i+1}) \in A$, we see that $f^n(\tilde{J}_i)$ is the convex hull of a basic path π_i which contains J. ■

LEMMA 4.5. *Suppose that $f\colon T \to T$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. Assume that J is an interval and π is a basic path such that $J \subset \langle \pi \rangle$ and $f(J)$ is contained in the convex hull of a basic path but is not reduced to a single point. Then there exists a unique basic path $\tilde{\pi}$ such that $f(J) \subset \langle \tilde{\pi} \rangle \subset \langle f(\pi) \rangle$.*

Proof. Since f is A-monotone, then $f(\langle \pi \rangle) = \langle f(\pi) \rangle$ is the union of convex hulls of basic paths. So set $\langle f(\pi) \rangle = \bigcup_{i=0}^{l} \langle \pi_i \rangle$ with $l \geqslant 1$. Since $\langle f(\pi) \rangle$ is an interval, the sets $\langle \pi_i \rangle$ have pairwise disjoint interiors. On the other hand $f(J) \subset f(\langle \pi \rangle) = \bigcup_{i=1}^{l} \langle \pi_i \rangle$ and the lemma follows. ■

Let $\{I_1, I_2, \ldots, I_k\}$ be the set of $A \cup V(T)$-basic intervals of T. Let $f\colon T \to T$ be a tree map such that the set $A \cup V(T)$ is f-invariant and f is $A \cup V(T)$-monotone. The $A \cup V(T)$-*graph of f* is the oriented graph whose vertices are in one-to-one correspondence with $I_1, I_2, \ldots, I_k$, and there is an oriented edge from I_i to I_j (by abuse of notaton) if and only if $f(I_i) \supset I_j$. Then $M_{\mathrm{M}}((T, A \cup V(T)), f))$ is the transition matrix of the $A \cup V(T)$-graph of f. By standard arguments (see for instance [8] or [1]), we get that

$$h(f) = \log \max\{\rho(M_{\mathrm{M}}((T, A \cup V(T)), f)), 1\}.$$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since f is A-monotone, we have that $f(A \cup V(T)) \subset A \cup V(T)$ and f is $A \cup V(T)$-monotone by Proposition 4.2. Set $M_{\mathrm{M}} = M_{\mathrm{M}}((T, A \cup V(T)), f)$ and $M_{\mathrm{P}} = M_{\mathrm{P}}(\mathscr{T}, \Theta)$. By Theorem 3.1, since $h(f) = \log \max\{\rho(M_{\mathrm{M}}), 1\}$, it is enough to show that

$$\rho(M_{\mathrm{M}}) \leqslant \rho(M_{\mathrm{P}})$$

when $\rho(M_{\mathrm{M}}) > 1$.

Let $n \in \mathbb{N}$ be such that the ith entry of the diagonal of M_{M}^n is equal to $m \geqslant 3$ (since $\rho(M_{\mathrm{M}}) > 1$, there always exists such an integer). This means that there are m different loops of length n in the $A \cup V(T)$-graph of f starting at I_i (here we use the notation from the definition of the matrix M_M). Let

$$I_i \to J_1 \to J_2 \to \cdots \to J_{n-1} \to I_i$$

be one of these loops. Set $K_0 = I_i$, $K_j \subset J_{n-j}$ a minimal interval such that $f(K_j) = K_{j-1}$ for $j = 1, 2, \dots, n-1$ and $K_n \subset I_i$ a minimal interval such that $f(K_n) = K_{n-1}$. Clearly K_n is an interval with the properties that $f^n(K_n) = I_i$ and $f^j(K_n) \subset J_i$ for $j = 1, 2, \dots, n-1$. We will say that the interval K_n is *associated* to the above loop. Let

$$I_i \to \hat{J}_1 \to \hat{J}_2 \to \cdots \to \hat{J}_{n-1} \to I_i$$

be another one of these loops different from the previous one, and let $\hat{K}_n$ be the associated interval. Clearly K_n and $\hat{K}_n$ have disjoint interiors. Therefore there are m intervals in I_i whose interiors are pairwise disjoint such that the image of each of them under f^n is I_i.

Let $\{\pi_1, \dots, \pi_s\}$ be the set of basic paths such that $\langle \pi_l \rangle \supset I_i$ for $l = 1, \dots, s$. By Lemma 4.4, I_i contains $m-2$ intervals $\tilde{J}_1, \dots, \tilde{J}_{m-2}$ whose interiors are pairwise disjoint, and to each $j \in \{1, \dots, m-2\}$ there corresponds a unique $l_j \in \{1, \dots, s\}$ such that $f^n(\tilde{J}_j) = \langle \pi_{l_j} \rangle \supset I_i$. Since $f^n(\mathrm{Int}(\tilde{J}_j)) = \mathrm{Int}(\langle \pi_{l_j} \rangle)$ if follows that $f^k(\tilde{J}_j)$ is contained in the convex hull of a basic path for $k = 0, 1, \dots, n$. From repeated use of Lemma 4.5, there is a unique loop

$$\pi_{l_j} \to \sigma_1 \to \sigma_2 \to \cdots \to \sigma_{n-1} \to \pi_{l_j},$$

of length n, in the $(\mathscr{T}, \Theta)$-path graph such that $f^k(\tilde{J}_j) \subset \langle \sigma_k \rangle$ for $k = 1, 2, \dots, n-1$ (recall that we also have $\tilde{J}_j \subset \langle \pi_{l_j} \rangle$ and $f^n(\tilde{J}_j) = \langle \pi_{l_j} \rangle$). We will say that such a loop is *associated* to $\tilde{J}_j$.

Let $j, l \in \{1, 2, \dots, m-2)$, where $j \neq l$. We claim that the loops associated to $\tilde{J}_j$ and $\tilde{J}_l$ are different. Otherwise, for $k = 0, 1, \dots, n$ the intervals $f^k(\tilde{J}_j)$ and $f^k(\tilde{J}_l)$ belong to the convex hull of the same basic path. Since f is A-monotone we get that $f^k|_K$ is monotone for each $k = 0, 1, \dots, n$, where $K = \langle \tilde{J}_j \cup \tilde{J}_l \rangle$. But this contradicts the fact that $f^n(\tilde{J}_j) = f^n(\tilde{J}_l) = \langle \pi_{l_j} \rangle$. This ends the proof of the claim.

For each $l \in \{1, 2, \dots, s\}$, let p_l be the cardinal of the set

$$\{j \in \{1, 2, \dots, m-2\} \colon l_j = l\}.$$

From above, for each $l \in \{1, 2, \dots, s\}$ there are at least p_l loops of length n in the $(\mathscr{T}, \Theta)$-path graph starting at π_l. If we consider the entry of the diagonal of the matrix M_P which corresponds to π_l then the $A \cup V(T)$-basic interval I_i contributes p_l to it (corresponding to those intervals $\tilde{J}_j \subset I_i$ such that $l_j = l$). Likewise, each $A \cup V(T)$-basic interval contained in $\langle \pi_l \rangle$ makes an analogous contribution. Since $\sum_{l=1}^{s} p_l = m-2$, for each $n \in \mathbb{N}$, we get that

$$\mathrm{Tr}(M_P^n) \geqslant \mathrm{Tr}(M_M^n) - 2v,$$

where v is the number of $A \cup V(T)$-basic intervals (that is, the size of M_M). Since M_M and M_P are non-negative matrices it follows that $|\mathrm{Tr}(M_P^n)| \geqslant |\mathrm{Tr}(M_M^n)| - 2v$ for each $n \in \mathbb{N}$. By [1, Lemma 4.4.2], we see that the sequence $\{|\mathrm{Tr}(M_M^n)|\}_{n \in \mathbb{N}}$ is unbounded, as $\rho(M_M) > 1$. So there exists n_0 such that $|\mathrm{Tr}(M_M^n)| > 2v$ for each $n > n_0$. Again by [1, Lemma 4.4.2], we obtain that

$$\begin{aligned} \rho(M_P) &= \limsup_{n \to \infty} |\mathrm{Tr}(M_P^n)|^{1/n} \geqslant \limsup_{n > n_0; n \to \infty} (|\mathrm{Tr}(M_M^n)| - 2v)^{1/n} \\ &= \limsup_{n \to \infty} |\mathrm{Tr}(M_M^n)|^{1/n} = \rho(M_M). \end{aligned}$$

This ends the proof of the theorem. ■

5. EXISTENCE OF A-MONOTONE MAPS AND PROOF OF THEOREM A

The following result demonstrates the existence of A-monotone maps exhibiting a given pattern.

THEOREM 5.1. *Let $(\mathcal{T}, \Theta)$ be a pattern. Then there exists a pointed tree (T, A) and a tree map $f: T \to T$ such that f exhibits $(\mathcal{T}, \Theta)$ over A and is A-monotone.*

Given this result, we are able to prove Theorem A.

Proof of Theorem A. By Theorem 5.1, we obtain (c). By Theorem 4.1 and statement (c), we have that $h(\mathcal{T}, \Theta) \leqslant \log \max\{\rho(M_P(\mathcal{T}, \Theta)), 1\}$. On the other hand, by Theorem 3.1, we have $h(\mathcal{T}, \Theta) \geqslant \log \max\{\rho(M_P(\mathcal{T}, \Theta)), 1\}$. This proves statement (a). Statement (b) follows from (a) and Theorem 4.1. ■

The rest of this section is devoted to proving Theorem 5.1. We split it into three subsections. In the first one we fix the notation and introduce some preliminary results. In the second one we perform the recursive construction of a tree given by the statement of Theorem 5.1. Finally, we construct an A-monotone map over this tree, and we prove Theorem 5.1.

5.1. Notation and preliminary results

Let $g: A \to S$ be a map from a finite set A to a tree S. We are going to construct a tree whose endpoints are identified with the points of A by an identification p. Let S' be the tree obtained from S by adding a new edge $\langle g(a), e_a \rangle$ for each point $a \in A$ so that if $a \neq b$ but $g(a) = g(b)$ then $e_a \neq e_b$. Let $r: S' \to S$ be the natural retraction from S' to S. That is, r is defined by

$$r(x) = \begin{cases} g(a) & \text{if } x \in \langle g(a), e_a \rangle \text{ for some } a \in A \\ x & \text{otherwise.} \end{cases}$$

Define a map $g': A \to S'$ by $g'(a) = e_a$. Let S_g be the tree $\langle g'(A) \rangle_{S'}$, and let $i: S_g \to S'$ denote the natural inclusion map. Define p to be the map $p: A \to S_g$ satisfying $i \circ p(a) = g'(a)$ for all $a \in A$. Then S_g is a tree satisfying $\mathrm{En}(S_g) = p(A)$, and it is unique up to homeomorphisms (see the following diagram).

$$\begin{tikzcd} S_g \arrow[r, "i"] & S' \arrow[r, "r"] & S \\ & A \arrow[ul, "p"] \arrow[u, "g'"] \arrow[ur, "g"'] & \end{tikzcd}$$

The map $r \circ i: S_g \to S$ will be called a *g-extension*. Notice that each of the above steps is well defined and unique (up to homeomorphisms). So the tree S_g and the g-extension are well defined. In what follows, we will forget about the identification p and consider $A \subset S_g$ in order to simplify the notation. The g-extension operation can be thought of as a global folding operation similar to that defined by Dicks [14] and Stallings [29].

To illustrate the notion of a g-extension consider Example 1.1 and take $g:\{x_1, x_2, x_3, x_5\} \to S$ such that $g = f|_{\{x_1,x_2,x_3,x_5\}}$ and $S = T$. Then S_g is a tree which is homeomorphic to $\langle x_2, x_3, x_4, x_6\rangle_T$ and the map $r \circ i$ (which is the g-extension) is just the homeomorphism between S_g and $\langle x_2, x_3, x_4, x_6\rangle_T$.

If $|A| = |g(A)|$ and $g(A) \subset \mathrm{En}(S)$ then the tree S_g is homeomorphic to $\langle g(A)\rangle_S$, and the g-extension is just the identity. In particular, if S is a star, $|A| = |g(A)|$ and $g(A) \subset \mathrm{En}(S)$ then S_g is a star. If on the other hand $|g(A)| = 1$ then S_g is also a star. The next lemma studies the trees S_g given by the restriction of g to a subset of A.

LEMMA 5.2. *Let $g: A \to S$ be a map from a finite set A to a tree S. Let $B \subset A$ and $g' = g|_B$. Then $S_{g'}$ is homeomorphic to $\langle B\rangle_{S_g}$. In particular, if $\langle g(B)\rangle_S$ is a $|B|$-star and $|B| = |g(B)|$ then $\langle B\rangle_{S_g}$ is also a $|B|$-star.*

Proof. We will denote the tree S' constructed from A (respectively B) by S'_A (respectively S'_B). Clearly, we can take S'_B to be a subtree of S'_A. Then $S_{g'}$ is a subtree of S_g and the first statement holds. The second statement follows from the first one and the definition of g-extension. ■

The next lemma displays the basic property of g-extensions.

LEMMA 5.3. *Let $g: A \to S$ be a map from a finite set A to a tree S. Then the g-extension is A-monotone.*

Proof. From the definition of S_g, it follows that $\mathrm{En}(S_g) = A$. Since i is the natural inclusion and r is a retraction we have that the g-extension is continuous. Moreover, the g-extension is defined by $r \circ i$, i.e. it is simply a restriction followed by a retraction, so it is clearly A-monotone. ■

Let (T, A) be a pointed tree. To simplify the notation in what follows, if (T, A) is a pointed tree we will not necessarily assume that $A \subset T$. In this setting, when writing (T, A) one should understand $(T, A \cap T)$. The set of discrete components of (T, A) will be denoted by $\mathscr{D}(T, A)$. Any subset Q of a discrete component of a pointed tree (T, A) such that either $|Q| = |A \cap T| = 1$ or $|Q| \geqslant 2$ will be called a *discrete subcomponent of* (T, A) (see Example 5.7).

Let $([T, A], [\theta])$ be a pattern, and let Q be a discrete subcomponent of (T, A). We will denote the set of discrete components of $(\langle\theta(Q)\rangle_T, A)$ by $\mathscr{S}(Q)$. This definition is independent of the chosen tree T in the class. If there is some $Q' \in \mathscr{S}(Q)$ such that $|Q'| = 1$ then $\mathscr{S}(Q) = \{Q'\}$. For each $Q' \in \mathscr{S}(Q)$ such that $|Q'| > 1$ there exists a unique $P \in \mathscr{D}(T, A)$ such that $Q' = \langle\theta(Q)\rangle_T \cap P$; in particular, Q' is a discrete subcomponent of (T, A). With this in mind, we define a *Q-sequence of the pattern* $([T, A], [\theta])$ to be an infinite sequence of the form $Q_0Q_1Q_2 \ldots$, where

(a) $Q_0 = Q$ is a discrete subcomponent of (T, A),
(b) $Q_n \in \mathscr{S}(Q_{n-1})$ for each $n \in \mathbb{N}$.

This notion and its use are illustrated in Example 5.7.

We stress the fact that the discrete components and the Q-sequences depend only on the given pattern but not on the chosen pointed tree. In particular, if

$([T, A], [\theta]) = ([T', A'], [\theta'])$ and $(T, A) \sim (T', A')$ are equivalent pointed trees by a bijection ϕ, then any Q-sequence obtained from (T, A) gives an equivalent $\phi(Q)$-sequence obtained from (T', A'). Therefore we will speak about discrete components and Q-sequences of a pattern with or without reference to a specific pointed tree.

Remark 5.4. Let (T, A) be any representative of $\mathscr{T}$, and let Q be a discrete subcomponent of (T, A). If for each discrete component Q' of $(\langle\theta(Q)\rangle_T, A)$, we replace the subtree $\langle Q'\rangle_T$ by any tree whose endpoints are Q' then we obtain another representative of $[\langle\theta(Q)\rangle_T, A]$, which in particular is a pointed tree.

The next lemma summarizes some properties of Q-sequences.

LEMMA 5.5. *Let $Q_0Q_1Q_2, \ldots$, be a Q-sequence of a pattern $(\mathscr{T}, \Theta)$.*

(a) $|Q_{n-1}| \geqslant |Q_n|$ *for each* $n \in \mathbb{N}$.
(b) *There exists* $m \in \mathbb{N} \cup \{0\}$ *such that* $|Q_m| = |Q_k|$ *for all* $k \geqslant m$.
(c) *If* $|Q_{n-1}| = |Q_n|$ *for some* $n \in \mathbb{N}$ *and if* $Q_0Q_1Q_2 \ldots Q_{n-1}Q'_n \ldots$ *is another Q-sequence such that* $Q'_n \neq Q_n$ *then* $|Q'_n| = 2$.
(d) *If* $|Q_{n-1}| = |Q_n|$ *for some* $n \in \mathbb{N}$ *and* $\tilde{Q}_0\tilde{Q}_1\tilde{Q}_2 \ldots$ *is a* $\tilde{Q}_0$*-sequence such that* $\tilde{Q}_i \subset Q_i$ *for all* $i \geqslant 0$, *then* $|\tilde{Q}_{n-1}| = |\tilde{Q}_n|$.

Proof. Set $(\mathscr{T}, \Theta) = ([T, A], [\theta])$ and fix $n \in \mathbb{N}$. Clearly, $|\theta(Q_{n-1})| \leqslant |Q_{n-1}|$. If $|Q_n| = 1$ then (a) follows directly. Assume that $|Q_n| > 1$. Let $R = \langle\theta(Q_{n-1})\rangle_T$, and let Q_n be a discrete component of the pointed tree (R, A). Let $C = \langle Q_n\rangle_R \backslash Q_n$. For each $q \in Q_n \subset R$ there exists $p \in Q_{n-1}$ such that $\theta(p)$ and q lie in the same connected component of $R\backslash C$. Assume that $q' \in Q_n$ with $q' \neq q$ and that $p' \in Q_{n-1}$ is such that $\theta(p')$ and q' lie in the same connected component of $R\backslash C$. Since R is a tree, then $p \neq p'$. Therefore $|Q_{n-1}| \geqslant |Q_n|$ and the first statement follows.

The second statement follows from the first one.

Now we prove (c). Suppose that $|Q_{n-1}| = |Q_n|$. If $|Q_n| = 1$ then there is nothing to prove because $\mathscr{S}(Q_{n-1}) = \{Q_n\}$ and there does not exist a Q-sequence $Q_0Q_1Q_2 \ldots Q_{n-1}Q'_n \ldots$ with $Q'_n \neq Q_n$. Assume that $|Q_n| > 1$. Then we also have that $|Q'_n| > 1$. Let C' be the connected component of $R\backslash C$ (where R and C are defined as in the proof of (a)) such that $Q'_n \cap C' \neq \emptyset$. Clearly, C' is a non-trivial subtree of R, and $Q_n \cap C'$ consists of a single point, q, say. Since $|Q_{n-1}| = |Q_n|$, from the proof of the first statement it follows that there exists a unique point $p \in Q_{n-1}$ such that $\theta(p) \in C'$. Since $R = \langle\theta(Q_{n-1})\rangle_T$, then $C' = \langle q, \theta(p)\rangle_R$, and as $\langle Q'_n\rangle_R \subset C'$ we have that $|Q'_n| = 2$ as required.

To prove (d), suppose that $|Q_{n-1}| = |Q_n|$. If $|Q_n| = 1$ then as above there is nothing to prove. Assume that $|Q_n| > 1$. With the notation of the proof of statement (a), recall that for each $q \in Q_n \subset R$ there exists a unique $p \in Q_{n-1}$ such that $\theta(p)$ and q lie in the same connected component of $R\backslash C$. In particular, this statement is satisfied for $q \in \tilde{Q}_n$ and $p \in \tilde{Q}_{n-1}$. Therefore (d) holds. ■

5.2. Construction of a minimal tree

To a pattern $(\mathscr{T}, \Theta)$, we will now associate a pointed tree, denoted $(\mathbb{T}_{(\mathscr{T}, \Theta)}, A)$, which we will call *minimal*. In order to define the tree $\mathbb{T}_{(\mathscr{T}, \Theta)}$ we will associate a tree $T(Q)$ to each discrete component Q of $(\mathscr{T}, \Theta)$. The tree $\mathbb{T}_{(\mathscr{T}, \Theta)}$ will be the union of these trees. The construction of the trees $T(Q)$ is inductive. Roughly speaking, the process is the following. Consider the set of Q-sequences (see Fig. 2), and let $QQ_{i_1} \ldots Q_{i_k} \ldots$ be one of these

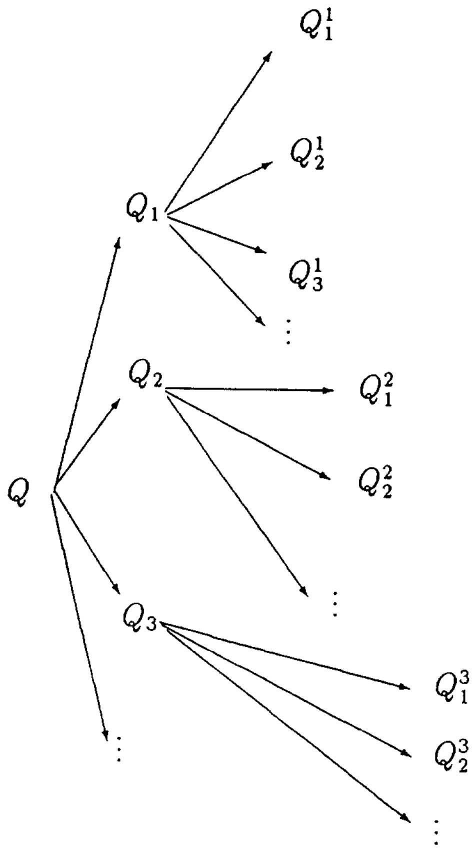

Fig. 2. The tree of all Q-sequences.

sequences. There is a first level n at which a Q-sequence starting with $QQ_{i_1} \ldots Q_{i_n}$ "stabilizes" in the sense of Lemma 5.5(b). We start the inductive process at level $n + 1$ by defining the trees $T(QQ_{i_1} \ldots Q_{i_{n+1}})$ to be $|Q_{i_{n+1}}|$-stars. Then we construct the trees $T(QQ_{i_1} \ldots Q_{i_n})$ by using the g-extension construction for a natural map g which will be defined. Finally, we check that this inductive process is well defined. The final tree is obtained by gluing together all the $T(Q)$. This final step is well defined because two discrete components P and Q have at most one point in common. Furthermore, the common point of $P \cup Q$ is an endpoint for both trees $T(P)$ and $T(Q)$. So the gluing operation mentioned above is just the identification of the common point of $T(P)$ and $T(Q)$.

Let $\mathbf{Q} = Q_0Q_1Q_2 \ldots$ be a Q-sequence of the pattern $(\mathscr{T}, \Theta)$. We define the *depth* of $\mathbf{Q}$, denoted by $\delta(\mathbf{Q})$, to be the minimum of all $n \in \mathbb{N} \cup \{0\}$ for which there exists a Q-sequence $Q_0Q_1Q_2 \ldots Q_nQ'_{n+1}Q'_{n+2} \ldots$ with $|Q'_j| = |Q_n|$ for all $j > n$. It is well defined by Lemma 5.5(b).

LEMMA 5.6. *The following statements hold.*

(a) *Let* $\mathbf{Q} = Q_0Q_1Q_2 \ldots$ *and* $\mathbf{Q}' = Q_0Q_1Q_2 \ldots Q_{\delta(\mathbf{Q})-1}Q'_{\delta(\mathbf{Q})} \ldots$ *be* Q-*sequences. Then* $\delta(\mathbf{Q}') \geqslant \delta(\mathbf{Q})$.
(b) *The set* $\{\delta(\mathbf{Q}): \mathbf{Q}$ *is a* Q-*sequence*$\}$ *is finite.*

Proof. The first statement follows directly from the definition of depth. To prove (b), let $\mathbf{Q} = Q_0Q_1Q_2 \ldots$ be a Q-sequence, and let M be the number of discrete subcomponents of the pattern plus the cardinality of A. So there exist $0 \leqslant i < j \leqslant M$ such that $Q_i = Q_j$. Then $Q_0Q_1Q_2 \dots Q_{i-1}(Q_i \dots Q_{j-1})^\infty$ is also a Q-sequence, and $\delta(\mathbf{Q}) \leqslant i < M$ by Lemma 5.5(a). ∎

Let $\underline{Q} = Q_0Q_1 \dots Q_n$ and $\underline{P} = P_0P_1 \dots P_m$ be two finite ordered sequences of discrete subcomponents. Formally, we will denote the sequence $Q_0Q_1 \dots Q_nP_0P_1 \dots P_m$ by $\underline{QP}$. For a sequence $\underline{Q} = Q_0Q_1 \dots Q_n$, the number n will be called the *length* of $\underline{Q}$ and will be denoted by $\|\underline{Q}\|$. We say that $\underline{Q}$ is *admissible* if there exists a Q_0-sequence starting with $\underline{Q}$ whose depth is larger than or equal to $\|\underline{Q}\| - 1$. If $\underline{Q}$ is *admissible* then the number $\delta(\underline{Q}) = \max\{\delta(\mathbf{Q})\colon \mathbf{Q} = \underline{Q} \dots\}$ will be called the *depth* of $\underline{Q}$. It is well defined by Lemma 5.6(b). By definition, $\delta(\underline{Q}) \geqslant \|\underline{Q}\| - 1$.

Now we may carry out the iterative construction of the tree $T(Q)$.

Step 0 (*Fill all depths*): For each admissible sequence $\underline{Q} = QQ_1Q_2 \dots Q_n$ for which $\delta(\underline{Q}) = \|\underline{Q}\| - 1$, we define $T(\underline{Q})$ to be a $|Q_n|$-star whose endpoints are Q_n (recall that a 1-star was defined to be a point).

Step k: Let $k \geqslant 0$. Assume by induction, that for all admissible seq uences $\underline{Q} = QQ_1Q_2 \dots Q_m$ with $\delta(\underline{Q}) \leqslant \|\underline{Q}\| - 1 + k$, we have defined the tree $T(\underline{Q})$ so that $\mathrm{En}(T(\underline{Q})) = Q_m$. By Step 0, this induction hypothesis is satisfied for $k = 0$.

Let $\underline{Q}' = QQ_1Q_2 \dots Q_n$ be an admissible sequence such that $\delta(\underline{Q}') = \|\underline{Q}'\| + k$. For each $Q^* \in \mathscr{S}(Q_n)$, the sequence $\underline{Q}'Q^*$ is admissible and satisfies $\delta(\underline{Q}'Q^*) \leqslant \delta(\underline{Q}') = \|\underline{Q}'\| + k = \|\underline{Q}'Q^*\| - 1 + k$. So each tree $T(\underline{Q}'Q^*)$ is defined by the induction hypothesis in such a way that $\mathrm{En}(T(\underline{Q}'Q^*)) = Q^*$.

In order to construct the tree $T(\underline{Q}')$, we first define an intermediate tree $R(\underline{Q}')$ in the following way:

$$R(\underline{Q}') = \bigcup_{Q^* \in \mathscr{S}(Q_n)} T(\underline{Q}'Q^*),$$

where the union is obtained as the disjoint union of the trees $T(\underline{Q}'Q^*)$ and the identification of the common points. By Remark 5.4, $(R(\underline{Q}'), A)$ is a pointed tree whose discrete components are $\mathscr{S}(Q_n)$. Then we apply the g-extension construct ion, where $g\colon Q_n \to R(\underline{Q}')$ is given by $g = \theta|_{Q_n}$ and we set $T(\underline{Q}') = S_g$. Then $T(\underline{Q}')$ is a tree whose endpoints are Q_n. This completes Step k of the induction process.

So for each admissible sequence $\underline{Q}$, we have defined the tree $T(\underline{Q})$. By carrying out the above iterative process, we obtain the trees $T(Q)$ for each discrete component Q of $(\mathscr{T}, \Theta)$. Finally we define $\mathbb{T}_{(\mathscr{T},\Theta)}$ to be $\bigcup_{Q \in \mathscr{Q}(\mathscr{T},\Theta)} T(Q)$, where the union is taken as above. By the definition of a pointed tree, it follows that $\mathbb{T}_{(\mathscr{T},\Theta)}$ is well defined and

$$([\mathbb{T}_{(\mathscr{T},\Theta)}, A], [\theta]) = (\mathscr{T}, \Theta).$$

Example 5.7. To illustrate the above notions we calculate explicitly the minimal tree for the pattern $([T, A], [\theta])$ with $\theta = f|_A$ as given in Example 1.1.

First, since any tree T with $|\mathrm{En}(T)| = k \leqslant 3$ is a k-star, our construction of $T(\underline{Q})$ for an admissible sequence $\underline{Q} = Q_0, \dots, Q_n$ with $|Q_n| \leqslant 3$ is easy. So $T(\underline{Q})$ must be a $|Q_n|$-star, and the tree $T(Q)$ corresponding to the discrete component $Q = \{x_4, x_5, x_6\}$ is a 3-star.

Now we construct the tree $T(Q)$ corresponding to the discrete component $Q = \{x_1, x_2, x_3, x_5\}$. Using the notation from the beginning of this subsection, we have $Q_1 = \{x_2, x_3, x_5\}$ and $Q_2 = \{x_5, x_4, x_6\}$. Thus $T(QQ_1)$ and $T(QQ_2)$ are 3-stars, and $R(Q)$ is homeomorphic to $\langle B\rangle_T$, where $B = \{x_2, x_3, x_5, x_4, x_6\}$. Since $T(Q)$ is defined by means of a g-extension of $g\colon Q \to R(Q)$, we obtain the tree shown in Fig. 3.

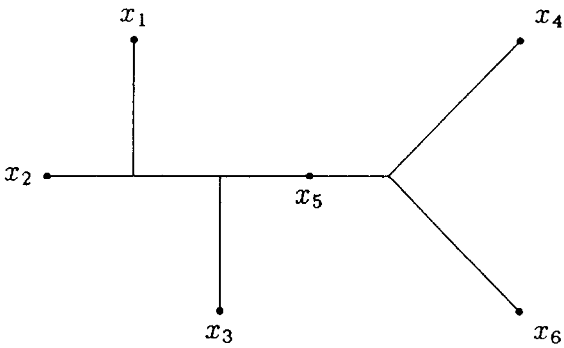

Fig. 3. The tree $\mathbb{T}_{(\mathcal{T},\Theta)}$ from Example 5.7.

5.3. Construction of an A-monotone map

Our first objective is to show that if a discrete subcomponent Q^* appears in a Q-sequence and in a Q'-sequence then the trees $T(Q \dots Q^*)$ and $T(Q' \dots Q^*)$ are homeomorphic. This is achieved in Lemma 5.10 and it will be essential for the proof of Theorem 5.11.

Let $\underline{Q} = Q_0Q_1 \dots Q_n$ be an admissible sequence with $n \geqslant 1$, and let $P_0P_1 \dots P_{n-1}$ be an admissible sequence so that P_0 is a discrete *component* of the pattern and $Q_i \subset P_{i-1}$ for $i = 1, 2, \dots, n$. Each P_i is determined uniquely, provided that $|Q_{i+1}| > 1$. Such an admissible sequence will be called *a dominant of* $\underline{Q}$.

LEMMA 5.8. *Let* $\underline{Q} = Q_0Q_1 \dots Q_n$ *be an admissible sequence, where* $n \geqslant 1$ *and* $\delta(\underline{Q}) = \|\underline{Q}\| - 1$. *Let* $\underline{P}$ *be a dominant of* $\underline{Q}$. *Then* $\langle Q_n \rangle_{T(\underline{P})}$ *is a* $|Q_n|$*-star.*

Proof. From Lemma 5.5(c), either $|Q_n| \in \{1, 2\}$ or $|Q_n| = |Q_{n-1}|$. If $|Q_n| \in \{1, 2\}$ then $\langle Q_n \rangle_T$ is a $|Q_n|$-star for any tree T and the statement holds. So assume that $|Q_n| = |Q_{n-1}|$. Consider the Q-sequence $\mathbf{Q} = Q_0Q_1 \dots Q_nQ_{n+1} \dots$ such that $|Q_i| = |Q_{n-1}|$ for all $i \geqslant n$ and the P-sequence $\mathbf{P} = \underline{P} \dots = P_0P_1 \dots P_{n-1}P_n \dots$ such that $Q_i \subset P_{i-1}$ for all $i \geqslant 1$. Set $\delta = \delta(\mathbf{P})$ and $\underline{P}' = P_0P_1 \dots P_\delta$. By Lemma 5.5(d), $\delta \geqslant \delta(\underline{Q}) - 1 = n - 2$. By construction $T(\underline{P}'P_{\delta+1})$ is a $|P_{\delta+1}|$-star. Since $P_{\delta+1}$ is a discrete subcomponent of $(\mathcal{T}, \Theta)$, $Q_{\delta+2} \subset P_{\delta+1}$ and $T(\underline{P}'P_{\delta+1})$ is a subtree of $R(\underline{P}')$, then $\langle Q_{\delta+2} \rangle_{T(\underline{P}'P_{\delta+1})} = \langle Q_{\delta+2} \rangle_{R(\underline{P}')}$ is a $|Q_{\delta+2}|$-star.

If $n = \delta + 2$ then we are done. Otherwise, since $|Q_{\delta+2}| = |Q_{\delta+1}|$ and $\langle Q_{\delta+2} \rangle_{R(\underline{P}')}$ is a subtree of $\langle \theta \langle Q_{\delta+1} \rangle \rangle_{R(\underline{P}')}$ then $\langle \theta \langle Q_{\delta+1} \rangle \rangle_{R(\underline{P}')}$ is a $|Q_{\delta+1}|$-star. Since $T(\underline{P}')$ is defined by means of a $\theta|_{P_\delta}$-extension then $\langle Q_{\delta+1} \rangle_{T(\underline{P}')} = \langle Q_{\delta+1} \rangle_{R(P_0P_1 \dots P_{\delta-1})}$ is a $|Q_{\delta+1}|$-star by Lemma 5.2. By repeating this argument $\delta + 1 - (n - 1)$ times we get the desired result. ■

LEMMA 5.9. *Let* $\underline{Q} = Q_0Q_1 \dots Q_n$ *be an admissible sequence, where* $\delta(\underline{Q}) \geqslant \|\underline{Q}\| \geqslant 1$, *and let* $\underline{P} = P_0P_1 \dots P_{n-1}$ *be a dominant of* $\underline{Q}$. *Suppose that* $T(\underline{Q}Q')$ *is homeomorphic to* $\langle Q' \rangle_{T(\underline{P}P')}$ *for each* $Q' \in \mathcal{S}(Q_n)$, *where* $P' \in \mathcal{S}(P_{n-1})$ *is such that* $Q' \subset P'$. *Then* $T(\underline{Q})$ *is homeomorphic to* $\langle Q_n \rangle_{T(\underline{P})}$.

Proof. Since $\delta(\underline{Q}) \geqslant n$ then, by Lemma 5.5(d), we obtain $\delta(\underline{P}) \geqslant \delta(\underline{Q}) - 1 \geqslant n - 1 = \|\underline{P}\|$. Following the previous subsection, we construct the trees $R(\underline{Q})$ and $R(\underline{P})$. By hypothesis, for each $Q' \in \mathcal{S}(Q_n)$, the tree $T(\underline{Q}Q')$ is homeomorphic to $\langle Q' \rangle_{T(\underline{P}P')}$, where $P' \in \mathcal{S}(P_{n-1})$. But $T(\underline{P}P')$ is a subtree of $R(\underline{P})$, and $Q' \subset P'$. Hence, $\langle Q' \rangle_{T(\underline{P}P')} = \langle Q' \rangle_{R(\underline{P})}$, so

$R(\underline{Q}) = \bigcup_{Q' \in \mathcal{S}(Q_n)} T(\underline{Q}Q')$ is homeomorphic to $\bigcup_{Q' \in \mathcal{S}(Q_n)} \langle Q' \rangle_{R(\underline{P})} = \langle \theta(Q_n) \rangle_{R(\underline{P})}$. In particular, $R(\underline{Q})$ is homeomorphic to a subtree of $R(\underline{P})$.

Let $g: P_{n-1} \to R(\underline{P})$, where $g = \theta|_{P_{n-1}}$. From the previous subsection we have that $S_g = T(\underline{P})$. Now consider $g|_{Q_n}: Q_n \to R(\underline{Q})$. Similarly we have that $S_{g|Q_n} = T(\underline{Q})$. But $R(\underline{Q})$ is homeomorphic to $\langle \theta(Q_n) \rangle_{R(\underline{P})}$, so by Lemma 5.2 and the construction of $T(\underline{P})$ it follows that $T(\underline{Q})$ is homeomorphic to $\langle Q_n \rangle_{T(\underline{P})}$ as required. ∎

LEMMA 5.10. *The tree $R(Q)$ is homeomorphic to $\langle \theta(Q) \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$ for each discrete component Q of $(\mathcal{T}, \Theta)$.*

Proof. Set $\underline{Q} = Q_0 = Q$. Clearly, $\delta(\underline{Q}) \geqslant \|\underline{Q}\| = 0$. Then $R(\underline{Q})$ is obtained as the union of the trees $T(Q_0Q_1)$, where the union is taken over all $Q_1 \in \mathcal{S}(Q_0)$. Also $\langle \theta(Q) \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$ is the union of the trees $\langle Q_1 \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$, where the union is taken over all $Q_1 \in \mathcal{S}(Q_0)$. Now we shall show that $T(Q_0Q_1)$ is homeomorphic to $\langle Q_1 \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$ for each $Q_1 \in \mathcal{S}(Q_0)$, and this will prove the lemma.

If $\delta(Q_0Q_1) = 0 = \|Q_0Q_1\| - 1$ then $T(Q_0Q_1)$ is a $|Q_1|$-star by definition. By Lemma 5.8, $\langle Q_1 \rangle_{T(P_0)}$ is a $|Q_1|$-star, where P_0 is the discrete component containing Q_1. This proves the statement in this case. If $\delta(Q_0Q_1) \geqslant 1 = \|Q_0Q_1\|$ then by Lemma 5.9 it is enough to show that $T(Q_0Q_1Q_2)$ is homeomorphic to $\langle Q_2 \rangle_{T(P_0P_1)}$ for each $Q_2 \in \mathcal{S}(Q_1)$, where $P_1 \in \mathcal{S}(P_0)$ is such that $Q_2 \subset P_1$.

From above, we only have to show that $T(\underline{Q})$ is homeomorphic to $\langle Q_n \rangle_{T(\underline{P})}$ for any admissible sequence $\underline{Q} = Q_0Q_1 \dots Q_n$ with $\delta(\underline{Q}) = \|\underline{Q}\| - 1$, where $\underline{P}$ is a dominant of $\underline{Q}$. This follows directly from the definition of $T(\underline{Q})$ (which is a $|Q_n|$-star) and Lemma 5.8. ∎

Now Theorem 5.1 follows immediately from the following result.

THEOREM 5.11. *Let $(\mathcal{T}, \Theta)$ be a pattern, and let $(\mathbb{T}_{(\mathcal{T},\Theta)}, A)$ be the associated minimal pointed tree. Then there exists $f: \mathbb{T}_{(\mathcal{T},\Theta)} \to \mathbb{T}_{(\mathcal{T},\Theta)}$, a continuous map, such that it exhibits $(\mathcal{T}, \Theta)$ over A and is A-monotone.*

Proof. Define $f: \mathbb{T}_{(\mathcal{T},\Theta)} \to \mathbb{T}_{(\mathcal{T},\Theta)}$ as follows. Take $\theta: A \to A$ so that $[\theta] = \Theta$. Fix $Q \in \mathcal{D}(\mathbb{T}_{(\mathcal{T},\Theta)}, A)$ and denote the $\theta|_Q$-extension from $T(Q) = \langle Q \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$ to $R(Q)$ by g_Q. By Lemma 5.10, we know that $R(Q)$ is homeomorphic to $\langle \theta(Q) \rangle_{\mathbb{T}_{(\mathcal{T},\Theta)}}$ by a homeomorphism φ_Q. By construction, φ_Q fixes $A \cap R(Q)$ pointwise. Then we set $f|_{T(Q)} = \varphi_Q \circ g_Q$.

Since the trees $T(Q)$ only intersect each other at their endpoints which are points of A and $f|_A = \theta$, f is well defined and continuous. Let us check that f is A-monotone. Since a map is A-monotone if and only if it is A-monotone restricted to the convex hull of each discrete component, it is enough to prove that $f|_{T(Q)}$ is A-monotone for each $Q \in \mathcal{D}(\mathbb{T}_{(\mathcal{T},\Theta)}, A)$. This follows directly from Lemma 5.3 and from the fact that φ_Q is a homeomorphism. ∎

6. CANONICAL MODELS

This section is devoted to proving Theorem B. We start with three technical lemmas.

As we indicated in Section 2, the following lemma gives us a better understanding of the equivalence of pointed trees.

LEMMA 6.1. *Let (T, A) and (T', A') be pointed trees and let $\phi: A \to A'$ be a bijection. Then the following statements are equivalent:*

(a) *ϕ preserves discrete components.*
(b) *π is a basic path of (T, A) if and only if $\phi(\pi)$ is a basic path of (T', A').*
(c) *For each $a, b, c \in A$ we have that $a \in \langle b, c\rangle_T$ if and only if $\phi(a) \in \langle \phi(b), \phi(c)\rangle_{T'}$.*

Proof. The fact that (a) and (b) are equivalent follows directly from the definitions of basic path and discrete component. Now we prove that (b) is equivalent to (c). If $|A| = |A'| < 3$ there is nothing to prove. So suppose that $|A| = |A'| \geqslant 3$. First, we prove that (c) implies (b). It is enough to show the "only if" part of (b). The "if" part can be proved by taking ϕ^{-1} instead of ϕ. Let π be a basic path of (T, A), and suppose that $\phi(\pi)$ is not. Then there exists a point $z \in A \backslash \pi$ such that $\phi(z) \in \langle \phi(\pi)\rangle$. By (c), we get that $z \in \langle \pi \rangle$; a contradiction. Now we prove that (b) implies (c). As above, it is enough to prove the "only if" part of (c). Assume that $a, b, c \in A$ satisfy $a \in \langle b, c\rangle$. If $\{b, a\}$ and $\{a, c\}$ are basic paths then they belong to different discrete components. Thus $\{\phi(b), \phi(a)\}$ and $\{\phi(a), \phi(c)\}$ are basic paths in different discrete components of (T', A'). So $\phi(a) \in \langle \phi(b), \phi(c)\rangle$. If $\{b, a\}$ or $\{a, c\}$ is not a basic path then the statement follows inductively either by considering $\langle a, b\rangle \cap A$ or $\langle a, c\rangle \cap A$ to be a finite union of basic paths. ■

Let $(\mathscr{T}, \Theta)$ be a pattern and let $f: T \to T$ be an A-monotone tree map which exhibits the pattern $(\mathscr{T}, \Theta)$ over A. By Proposition 4.2 and the definition of identifiable vertices we easily obtain the following result.

LEMMA 6.2. *The f-identifiability relation is an equivalence relation and satisfies the following properties.*

(a) *If v_1 and v_2 are f-identifiable then either $f^n(v_1), f^n(v_2) \in V(T) \backslash A$ are f-identifiable for all $n \geqslant 0$, or there exists $n \geqslant 1$ such that $f^n(v_1) = f^n(v_2) \in A$ and $f^i(v_1), f^i(v_2) \in V(T)\backslash A$ are f-identifiable for all $i < n$.*
(b) *If v_1 and v_2 are f-identifiable and $v_3 \in \langle v_1, v_2\rangle \cap V(T)$ then v_1, v_2 and v_3 are pairwise f-identifiable.*

Let (T, A) and (T', A') be two equivalent pointed trees. Then there exists a bijection $\phi: A \to A'$ which preserves discrete components of the pointed trees (T, A) and (T', A'). For ease of reading, in the rest of the section we will omit the map ϕ, and we will identify the sets A and A'. Given two points x and y of a tree T, we denote the number of vertices of T in the interior of $\langle x, y\rangle_T$ by $\nu(\{x, y\}, T)$. Two vertices of T will be called *adjacent* if they lie on the same edge.

The following lemma will be necessary for the proof of part (c) of Theorem B.

LEMMA 6.3. *Let (T, A) and (T', A) be equivalent pointed trees such that $\mathrm{En}(T) \cup \mathrm{En}(T') \subset A$. Then there exists a homeomorphism $h: T \to T'$ such that $h|_A = \mathrm{Id}|_A$ if and only if $\nu(\pi, T) = \nu(\pi, T')$ for each basic path π of (T, A) and (T', A).*

Proof. The "only if" part is trivial. So assume that (T, A) and (T', A) are pointed trees such that $\mathrm{En}(T)$ and $\mathrm{En}(T')$ are contained in A and $\nu(\pi, T) = \nu(\pi, T')$ for each basic path π. The homeomorphism h will be constructed by defining h on the convex hull of each discrete component of (T, A). If a discrete component has cardinality $n \leqslant 3$ then the definition of h is obvious because each tree with n endpoints is an n-star. So suppose that Q is a discrete component whose cardinality is at least four.

Let π_1 and π_2 be two different basic paths of Q such that $\langle \pi_1 \rangle_T \cap \langle \pi_2 \rangle_T \neq \emptyset$. Then $S = \langle \pi_1 \cup \pi_2 \rangle_T$ and $S' = \langle \pi_1 \cup \pi_2 \rangle_{T'}$ are trees with the same number $e \in \{3, 4\}$ of endpoints. Denote the endpoints of S and S' by $v_1, \dots, v_e$, and assume that $\pi_1 = \{v_1, v_2\}$ and that $v_1 \notin \pi_2$. Let w_1, w_2 (respectively w'_1, w'_2) be the vertices of S (respectively S') of valence at least three (with this notation we will have $w_1 = w_2$ (respectively $w'_1 = w'_2$) when S (respectively S') is a star).

For each $i = 1, \dots, e$, let $u_i \in \{w_1, w_2\}$ (respectively $u'_i \in \{w'_1, w'_2\}$) be the vertex of S (respectively S') adjacent (in S (respectively in S')) to v_i. By relabelling if necessary we may assume that $u_1 = u_3 = w_1$ (and $u_2 = u_4 = w_2$ when $e = 4$). For each $i = 1, \dots, e$, define $p(i) = \nu(\{v_i, u_i\}, T)$ and $p'(i) = \nu(\{v_i, u'_i\}, T')$.

We claim that $p(i) = p'(i)$ for each $i = 1, \dots, e$. We will prove the claim in the case $e = 4$; the proof in the other case is similar. Define $p = \nu(\{w_1, w_2\}, T) + \mathrm{Card}(\{w_1, w_2\})$ and $p' = \nu(\{w'_1, w'_2\}, T') + \mathrm{Card}(\{w'_1, w'_2\})$. Note that $p, p' \geqslant 1$. There are three cases to be considered.

Case (i): $u'_1 = u'_3$ *and* $u'_2 = u'_4$. By hypothesis, $\nu(\{v_i, v_j\}, T) = \nu(\{v_i, v_j\}, T')$ for all $i, j \in \{1, 2, 3, 4\}$. By considering $\langle v_1, v_2 \rangle$, $\langle v_1, v_3 \rangle$ and $\langle v_2, v_3 \rangle$, we see that

$$p(1) + p + p(2) = p'(1) + p' + p'(2)$$

$$p(1) + p(3) + 1 = p'(1) + p'(3) + 1$$

$$p(2) + p + p(3) = p'(2) + p' + p'(3).$$

Therefore $p(1) = p'(1)$ and $p(3) = p'(3)$. Symmetrically, by considering $\langle v_1, v_2 \rangle$, $\langle v_2, v_4 \rangle$ and $\langle v_1, v_4 \rangle$, it follows that $p(2) = p'(2)$ and $p(4) = p'(4)$. Thus the claim holds in this case.

Case (ii): $u'_1 = u'_4$ *and* $u'_2 = u'_3$. By considering $\langle v_1, v_2 \rangle$,$\langle v_1, v_3 \rangle$ and $\langle v_2, v_3 \rangle$, we see that

$$p(1) + p + p(2) = p'(1) + p' + p'(2)$$

$$p(1) + p(3) + 1 = p'(1) + p' + p'(3)$$

$$p(2) + p + p(3) = p'(2) + p'(3) + 1.$$

Then $p(3) = p'(3)$ and $p(1) = p'(1) + p' - 1$. Symmetrically, by considering $\langle v_3, v_4 \rangle$, $\langle v_3, v_1 \rangle$ and $\langle v_1, v_4 \rangle$, it follows that $p(3) = p'(3) + p' - 1$ and $p(1) = p'(1)$. Therefore $p' = 1$, and hence $p(2) + p - 1 = p'(2)$ from the first equation. Symmetrically we see that $p(4) + p - 1 = p'(4)$. Finally, by looking at the basic path $\langle v_2, v_4 \rangle$, we see that $p = 1$. So the claim is proved in this case.

Case (iii): $u'_1 = u'_2$ *and* $u'_3 = u'_4$. By considering $\langle v_1, v_2 \rangle$, $\langle v_1, v_3 \rangle$ and $\langle v_2, v_3 \rangle$, we see that

$$p(1) + p + p(2) = p'(1) + p'(2) + 1$$

$$p(1) + p(3) + 1 = p'(1) + p' + p'(3)$$

$$p(2) + p + p(3) = p'(2) + p' + p'(3).$$

Then $p(1) = p'(1)$ and $p(3) = p'(3) + p' - 1$. Symmetrically, by considering $\langle v_3, v_4 \rangle$, $\langle v_3, v_1 \rangle$ and $\langle v_1, v_4 \rangle$, it follows that $p(1) = p'(1) + p' - 1$ and $p(3) = p'(3)$. Therefore $p' = 1$, and hence $p(2) + p - 1 = p'(2)$ from the third equation. Then the claim follows as in the previous case. This ends the proof of the claim.

Next we construct the desired homeomorphism between T and T'.

In the rest of the proof we will consider all basic paths to be ordered. In other words, when we write $\{a, b\}$, we will assume that $a < b$. Let v be a vertex in the interior of $\langle a, b \rangle_T$ (respectively $\langle a, b \rangle_{T'}$). Denote the cardinal of the set $\langle a, v \rangle_T \cap V(T) \backslash \{a\}$ (respectively $\langle a, v \rangle_{T'} \cap V(T') \backslash \{a\}$) by $\varphi_{\{a,b\}}(v)$ (respectively $\varphi'_{\{a,b\}}(v)$).

Now fix a basic path $\pi = \{a, b\}$, denote the set of vertices in the interior of $\langle a, b \rangle_T$ and $\langle a, b \rangle_{T'}$ by $\{v_1, \dots, v_n\}$ and $\{v'_1, \dots, v'_n\}$ respectively. Both sets have the same cardinality because of the assumption. We will choose the labels of these vertices so that $\varphi_{\{a,b\}}(v_i) = \varphi'_{\{a,b\}}(v'_i) = i$. Let $h_\pi : \langle \pi \rangle_T \to \langle \pi \rangle_{T'}$ be a homeomorphism such that $h|_{\{a,b\}} = \mathrm{Id}|_{\{a,b\}}$ and $h_\pi(v_i) = v'_i$.

Let $\pi^* = \{a^*, b^*\}$ be a basic path such that $\langle \pi \rangle_T \cap \langle \pi^* \rangle_T \neq \emptyset$. Choose a vertex $v_i \in \langle \pi \rangle_T \cap \langle \pi^* \rangle_T$. From the above claim it follows that $v'_i \in \langle \pi^* \rangle_{T'}$, and $\varphi_{\{a^*,b^*\}}(v_i) = \varphi'_{\{a^*,b^*\}}(v'_i)$. Therefore we can choose the homeomorphisms h_π in such a way that if π and π^* are basic paths with $\langle \pi \rangle_T \cap \langle \pi^* \rangle_T \neq \emptyset$ then $h_\pi|_{\langle \pi \rangle_T \cap \langle \pi^* \rangle_T} = h_{\pi^*}|_{\langle \pi \rangle_T \cap \langle \pi^* \rangle_T}$. Finally, we define $h: T \to T'$ by $h(x) = h_\pi(x)$, where π is a basic path such that $x \in \langle \pi \rangle_T$. Clearly h is a well defined homeomorphism such that $h|_A = \mathrm{Id}|_A$. ■

Proof of Theorem B. We start by proving (a). By Theorem A, we know that there exists a tree map $f: T \to T$ which exhibits the pattern $(\mathscr{T}, \Theta)$ over A and is A-monotone. We will show that we can obtain a canonical model of $(\mathscr{T}, \Theta)$ from this monotone model. Indeed, let $\tilde{T}$ be the tree obtained by contracting the convex hull of the points in each class of the f-identifiability relation to a point. Each of these classes is contained in a connected component of $T \backslash A$. Therefore its convex hull is also contained in the same connected component of $T \backslash A$. Consequently $(\tilde{T}, A)$ is a pointed tree equivalent to (T, A). Let $\varphi: T \to \tilde{T}$ be the standard projection. So φ is continuous, injective in a neighbourhood of each point which does not belong to the convex hull of a class of the f-identifiability relation, and the image of each point in a convex hull C of class of the f-identifiability relation is the point to which C is contracted. Then we define $\tilde{f}: \tilde{T} \to \tilde{T}$ by $\tilde{f}(x) = \varphi \circ f(\tilde{x})$, where $\tilde{x} \in \varphi^{-1}(x)$. By virtue of the definition of φ and Lemma 6.2, the map $\tilde{f}$ is well defined and continuous. Since f is A-monotone, from the definition of φ, $\tilde{f}$ is also A-monotone. Clearly, $\tilde{f}$ exhibits the pattern $(\mathscr{T}, \Theta)$ over A. Furthermore, if v, $w \in V(\tilde{T}) \backslash A$ and are $\tilde{f}$-identifiable then $v = w$.

Statement (b) follows from Proposition 4.2 and from the fact that any canonical model is A-monotone.

Now we prove (c). Let (T, A, f) and (T', A, f') be two canonical models of the pattern $(\mathscr{T}, \Theta)$. Suppose that there is no homeomorphism $h: T \to T'$ such that $h|_A = \mathrm{Id}|_A$ and $f' \circ h|_A = h \circ f|_A$. By Lemma 6.3, there exists a basic path $\pi = \{a, b\}$ such that $v(\pi, T) \neq v(\pi, T')$. This will lead us to a contradiction. Denote the unique discrete component which contains π by Q. Then $|Q| \geqslant 4$.

We claim that there exist distinct points $z_1, z_2, z_3, z_4 \in Q$ such that $\langle z_1, z_2 \rangle_{T'} \cap \langle z_3, z_4 \rangle_{T'} = \emptyset$ but $\langle z_1, z_2 \rangle_T \cap \langle z_3, z_4 \rangle_T \neq \emptyset$. To prove the claim, define $\{v_1, \dots, v_n\} = \langle \pi \rangle_T \cap (V(T) \backslash A)$ by $\langle a, v_i \rangle_T \cap (V(T) \backslash A) = \{v_1, \dots, v_i\}$ for each $i = 1, \dots, n$, and define $\{w_1, \dots, w_m\} = \langle \pi \rangle_{T'} \cap (V(T') \backslash A)$ similarly. For each $i = 1, \dots, n$, set

$$v_i^T = \{x \in Q: \langle x, v_i \rangle_T \cap \langle \pi \rangle_T = \{v_i\}\}.$$

Analogously, for each $i = 1, \dots, m$, set

$$w_i^{T'} = \{x \in Q: \langle x, w_i \rangle_{T'} \cap \langle \pi \rangle_{T'} = \{w_i\}\}.$$

Then $\{v_i^T : i = 1, 2, \dots, n\}$ and $\{w_i^{T'}: i = 1, 2, \dots, m\}$ are partitions of $Q \setminus \{a, b\}$. Since $n = \nu(\pi, T) \neq \nu(\pi, T') = m$, there exists a least integer j such that $v_j^T \neq w_j^{T'}$. By interchanging T and T' if necessary, we may suppose that there exists $x \in v_j^T$ such that $x \in w_k^{T'}$, with $k > j$. Now take $y \in w_j^{T'}$. Then $y \in v_l^T$, where $l \geq j$, by the minimality of j. Since $w_j \neq w_k$, we see that $\langle a, y \rangle_{T'} \cap \langle x, b \rangle_{T'} = \emptyset$. So the claim holds by taking $z_1 = a$, $z_2 = y$, $z_3 = b$ and $z_4 = x$ (see Fig. 4).

From the claim, we can define $I = \langle r_1, r_2 \rangle_T = \langle z_1, z_2 \rangle_T \cap \langle z_3, z_4 \rangle_T$ and $I' = \langle r_1', r_2' \rangle_{T'}$ to be the closure of $\langle z_1, z_2, z_3, z_4 \rangle_{T'} \setminus (\langle z_1, z_2 \rangle_{T'} \cup \langle z_3, z_4 \rangle_{T'})$. Then I' is a non-degenerate interval, while I may degenerate to a point.

Since (T, A, f) and (T', A, f') are canonical models of the pattern $(\mathscr{T}, \Theta)$, there exists a least integer $i \geq 1$ such that $f^s(I) \cap A = \emptyset$, $f'^s(I') \cap A = \emptyset$ for each $0 \leq s < i$, and either:

(i) $f^i(I) \cap A \neq \emptyset$ and $f^i(r_1) \neq f^i(r_2)$ if $r_1 \neq r_2$, or
(ii) $f'^i(I') \cap A \neq \emptyset$ and $f'^i(r_1') \neq f'^i(r_2')$.

Since f and f' are A-monotone, then

$$f^s(I) = \langle f^s(r_1), f^s(r_2) \rangle_T \quad \text{and} \quad f'^s(I') = \langle f'^s(r_1'), f'^s(r_2') \rangle_{T'}$$

for each $0 \leq s \leq i$. Since (T, A, f) is a canonical model, if $r_1 \neq r_2$ then $f^s(r_1) \neq f^s(r_2)$ for each $0 \leq s \leq i$.

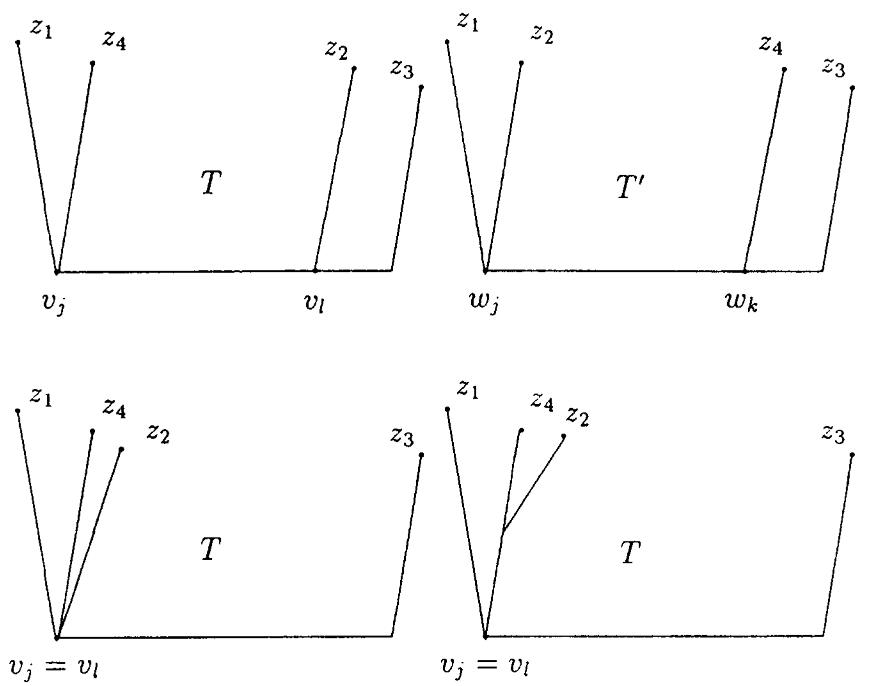

Fig. 4. The possible positions of the points $z_1, \dots, z_4$ in T and T'.

For each $0 \leqslant s < i$, we claim that there exists a discrete subcomponent $Q_s = \{z_j^s: j = 1, \ldots, 4\}$ such that

(a) $f^s(I) \subset \langle Q_s \rangle_T$ (respectively $f'^s(I') \subset \langle Q_s \rangle_{T'}$ and is a non-degenerate interval) and $f^s(\{r_1, r_2\})$ (respectively $f'^s(\{r'_1, r'_2\})$) are the vertices of $\langle Q_s \rangle_T$ of valence at least 3 (respectively of $\langle Q_s \rangle_{T'}$ of valence 3).
(b) There exist homeomorphisms $h_s: \langle Q_s \rangle_T \to \langle Q_0 \rangle_T$ and $h'_s: \langle Q_s \rangle_{T'} \to \langle Q_0 \rangle_{T'}$ such that $h_s(z_j^s) = h'_s(z_j^s) = z_j^0 = z_j$ for $j = 1, \ldots, 4$.

The claim is clearly true for $s = 0$ by taking $z_j^0 = z_j$ for $j = 1, \ldots, 4$. Let $0 < s < i$ and suppose that the claim is true for $s - 1$. Since f is A-monotone, either the tree $\langle f(Q_{s-1}) \rangle_T$ is homeomorphic to $\langle Q_{s-1} \rangle_T$ by a homeomorphism which maps $f(z_j^{s-1})$ to z_j^{s-1} for $j = 1, \ldots, 4$ and has $f^s(\{r_1, r_2\})$ as the vertices of valence at least 3, or there exist $j \in \{1, \ldots, 4\}$ and $p \in \{1, 2\}$ such that $f(z_j^{s-1}) = f^s(r_p)$ (recall that $f^s(r_1) \neq f^s(r_2)$ when $r_1 \neq r_2$). In the second case, since $z_j^{s-1} \in Q_{s-1} \subset A$ we have that $f^s(r_p) \in A$ which contradicts the minimality of i. In a similar way we conclude that the tree $\langle f'(Q_{s-1}) \rangle_{T'}$ is homeomorphic to $\langle Q_{s-1} \rangle_{T'}$ by a homeomorphism which maps $f'(z_j^{s-1})$ to z_j^{s-1} for $j = 1, \ldots, 4$ and has $f'^s(\{r'_1, r'_2\})$ as the vertices of valence 3 (recall that I' was a non-degenerate interval and that $f'^s(r'_1) \neq f'^s(r'_2)$ by (ii)).

Fix $j \in \{1, \ldots, 4\}$, and consider $p \in \{1, 2\}$ such that $A \cap \langle r_p, z_j^0 \rangle_T = \{z_j^0\}$. Since f is A-monotone, then $f^s(r_k) \notin \langle f^s(r_p), f(z_j^{s-1}) \rangle_T$ for $k \in \{1, 2\}$, $k \neq p$. Define z_j^s to be the element of $A \cap \langle f^s(r_p), f(z_j^{s-1}) \rangle_T$ which satisfies $A \cap \langle f^s(r_p), z_j^s \rangle_T = \{z_j^s\}$ (thus z_j^s is the point of A between $f^s(r_p)$ and $f(z_j^{s-1})$ closest to $f^s(r_p)$). From above we see that $z_j^s \in \langle f(z_j^{s-1}), f(z_k^{s-1}) \rangle_T$ for each $k \in \{1, 2, 3, 4\}$ and $k \neq j$. Since $f|_A = f'|_A$ and $(T, A) \sim (T', A)$, it follows from Lemma 6.1 that $z_j^s \in \langle f'(z_j^{s-1}), f'(z_k^{s-1}) \rangle_{T'}$ for each $k \in \{1, 2, 3, 4\}$ and $k \neq j$. So there exists $p' \in \{1, 2\}$ such that $f'^s(r'_{p'})$ is the vertex of valence 3 of $\langle f'(Q_{s-1}) \rangle_{T'}$ closest to $f'(z_j^{s-1})$, and z_j^s is the point of A between $f'^s(r'_{p'})$ and $f'(z_j^{s-1})$ closest to $f'^s(r'_{p'})$). Now statement (a) of the claim holds using (i) and (ii). Statement (b) follows by defining h_s and h'_s in an appropriate way. So the claim is proved.

Now consider the trees $\langle f(Q_{i-1}) \rangle_T \supset f^i(I)$ and $\langle f'(Q_{i-1}) \rangle_{T'} \supset f'^i(I')$. Then either $f^i(I) \cap A \neq \emptyset$ or $f'^i(I') \cap A \neq \emptyset$. Suppose first that $f^i(I) \cap A \neq \emptyset$ and take $u \in f^i(I) \cap A$. By the above two claims, $u \in \langle f(z_1^{i-1}), f(z_2^{i-1}) \rangle_T \cap \langle f(z_3^{i-1}), f(z_4^{i-1}) \rangle_T$, so by Lemma 6.1(c),

$$u \in \langle f'(z_1^{i-1}), f'(z_2^{i-1}) \rangle_{T'} \cap \langle f'(z_3^{i-1}), f'(z_4^{i-1}) \rangle_{T'}.$$

Since f' is A-monotone, there exists $u'_1 \in \langle z_1^{i-1}, z_2^{i-1} \rangle_{T'}$ and $u'_2 \in \langle z_3^{i-1}, z_4^{i-1} \rangle_{T'}$ such that $f'(u'_1) = f'(u'_2) = u$, and thus $f'(\langle u'_1, u'_2 \rangle_{T'}) = \{u\}$. Now, since $\langle z_1^{i-1}, z_2^{i-1} \rangle_{T'} \cap \langle z_3^{i-1}, z_4^{i-1} \rangle_{T'} = \emptyset$, in view of (a) of the second claim, we see that $f'^{i-1}(I') \subset \langle u'_1, u'_2 \rangle_{T'}$. Hence $f'^i(r'_1) = f'^i(r'_2) = u \in A$ — this contradicts the supposition that (T', A, f') is a canonical model. If on the other hand $f^i(I) \cap A = \emptyset$ then $f'^i(I') \cap A \neq \emptyset$. By the same reasoning, we deduce that there exist $u_1 \in \langle z_1^{i-1}, z_3^{i-1} \rangle_T$ and $u_2 \in \langle z_2^{i-1}, z_4^{i-1} \rangle_T$ satisfying $f(\langle u_1, u_2 \rangle_T) = \{u\}$. From (a) of the second claim, all vertices of valence at least 3 of $\langle Q_{i-1} \rangle_T$ are contained in $f^{i-1}(I)$, hence independently of the labels of the endpoints of Q_{i-1}, there exists a point $a \in f^{i-1}(I) \cap \langle u_1, u_2 \rangle_T$. So $u = f(a) \in f^i(I)$. This implies that $f^i(I) \cap A \neq \emptyset$ — a contradiction. This ends the proof of the first statement of (c).

The second statement of (c) follows from above and Proposition 4.2. ■

7. A-MONOTONE MODELS AND MINIMAL DYNAMICS

The goal of this section is to prove Theorem C and D. We start with three technical results.

LEMMA 7.1. *Let $f: T \to T$ be a tree map which exhibits the pattern $(\mathcal{T}, \Theta)$ over A and is A-monotone. Let x be a significant periodic point with period n. Then for each $k \geq 1$ there exists a unique basic path π such that $\langle \pi \rangle$ contains an interval J satisfying $x \in J$ and $f^{kn}(J) = \langle \pi \rangle$. Moreover, for each $i \leq kn$, $f^i(J)$ is contained in the convex hull of a basic path.*

Proof. Let $B = \bigcup_{i=0}^{kn} f^{-i}(A)$. Since x is significant, $x \notin B \cup V(T)$ (otherwise, since x is periodic of period n, $x \in A \cup V(T)$; a contradiction). We claim that if there exists a pair (π, J) satisfying the statement of the lemma, then it is unique. To prove the claim assume that there exist two pairs (π, J) and (π', J') with $\pi \neq \pi'$ satisfying the statement of the lemma. Then, since f is A-monotone, we see that $J \cap J'$ contains a point $y \in V(T)$. Note that $x \notin V(T)$ and so, $x \neq y$. Therefore, again by the A-monotonicity of f we see that $f^{kn}(y) = y$. So, x is f-monotone equivalent to y which contradicts the fact that x is significant. This ends the proof of the claim. Now we prove the existence of such a pair (π, J).

Since $x \notin B \cup V(T)$, there exist two elements $a, b \in B \cup V(T)$ such that $x \in \langle a, b \rangle \setminus \{a, b\}$ and there are no elements of $B \cup V(T)$ in the interior of $\langle a, b \rangle$. Consequently $f^{kn}|_{\langle a, b \rangle}$ is monotone.

Suppose first that $a, b \in B$. Then $f^{kn}(a), f^{kn}(b) \in A$, and since $f^{kn}|_{\langle a, b \rangle}$ is monotone, by the definition of a and b we obtain that $\{f^{kn}(a), f^{kn}(b)\}$ is a basic path. Denote this basic path by π. Since x is periodic of period n we have $x \in \langle a, b \rangle \cap \langle \pi \rangle$. So $\langle \pi \rangle \supset \langle a, b \rangle = J$. Moreover, by the definition of a and b we get that $f^i(J)$ is contained in the convex hull of a basic path for each $i \leq kn$. This ends the proof of the lemma in this case.

Now suppose that $\{a, b\} \not\subset B$. We are going to construct two finite sequences of points $\{a_i\}_{i=0}^{l}$ and $\{b_i\}_{i=0}^{l}$ such that

(1) $a_0 = a$ and $b_0 = b$,
(2) $\{a_i\}_{i=0}^{l} \cup \{b_i\}_{i=0}^{l} \subset B \cup V(T)$,
(3) $\langle a_i, b_i \rangle$ strictly contains $\langle a_{i-1}, b_{i-1} \rangle$ for $i = 1, 2, \dots, l$,
(4) the interior of $\langle a_i, b_i \rangle$ does not contain any element of B and the interior of $\langle a_i, b_i \rangle \setminus \langle a_{i-1}, b_{i-1} \rangle$ does not intersect $V(T)$ for $i = 1, 2, \dots, l$,
(5) $f^{kn}|_{\langle a_i, b_i \rangle}$ is monotone for each $i = 0, 1, \dots, l$,
(6) $\langle a_i, b_i \rangle$ is strictly contained in $f^{kn}(\langle a_i, b_i \rangle)$ for each $i = 0, 1, \dots, l-1$
(7) $\{a_i, b_i\} \not\subset B$ for $i = 0, 1, \dots, l-1$, and
(8) $a_l, b_l \in B$.

Then the proof of the lemma follows as above by using a_l and b_l instead of a and b.

Now let us construct recursively the above sequences. In view of (1) we start with $a_0 = a$ and $b_0 = b$. To see that properties (1)–(7) hold for $i = 0$ it only remains to show that $\langle a_0, b_0 \rangle$ is strictly contained in $f^{kn}(\langle a_0, b_0 \rangle)$. By (2) and Proposition 4.2 we have that $f^{kn}(a_0), f^{kn}(b_0) \in A \cup V(T)$. Since $f^{kn}(x) = x$, we have that $\langle a_0, b_0 \rangle \subset f^{kn}(\langle a_0, b_0 \rangle)$ by (5) and the definition of a and b. Now suppose that $\langle a_0, b_0 \rangle = f^{kn}(\langle a_0, b_0 \rangle)$. Then $f^{kn}(\{a_0, b_0\}) = \{a_0, b_0\}$ and hence $a_0, b_0 \in A \cup V(T)$. So x is f-monotone equivalent to a and b by (5); a contradiction with the fact that x is significant. Thus (6) also holds for $i = 0$.

Now suppose that we already have defined a_i and b_i for some $i \geqslant 0$ in such a way that properties (2)–(6) hold for all a_j, b_j with $j \leqslant i$ and (7) holds for $j \leqslant i-1$. If $a_i, b_i \in B$ then we set $l = i$ and we are done. Otherwise, (7) also holds and we define a_{i+1}, b_{i+1}. We only will define a_{i+1}; the point b_{i+1} is defined in a similar way. If $a_i \in B$ then we set $a_{i+1} = a_i$. Otherwise, $a_i \in V(T) \backslash B$. Denote the union of the closures of all connected components of $T \backslash \langle a_i, b_i \rangle$ which contain a_i (respectively b_i) by C_a (respectively C_b). By (5), $f^{kn}|_{\langle a_i, x \rangle}$ is monotone, and by (6), $f^{kn}(a_i) \in C_a \cup C_b$. Now we consider two cases.

Case 1: *Assume that* $f^{kn}(a_i) \in C_a$. Then $f^{kn}(a_i) \neq a_i$ because otherwise x will be f-monotone equivalent to a_i which belongs to $V(T)$; a contradiction with the fact that x is significant. Now we take $a_{i+1} \in (\langle f^{kn}(a_i), a_i \rangle \backslash \{a_i\}) \cap (B \cup V(T))$ so that $\langle a_{i+1}, a_i \rangle \cap (B \cap V(T)) = \{a_{i+1}, a_i\}$. This a_{i+1} exists because $f^{kn}(a_i) \in A \cup V(T)$, by Proposition 4.2. Moreover, since $a_i \notin B$, then $f^{kn}|_{\langle a_{i+1}, b_i \rangle}$ is monotone.

Case 2: *Assume that* $f^{kn}(a_i) \in C_b$. Consider the interval $I = \langle b_i, f^{kn}(a_i) \rangle$. If $I \cap B \neq \emptyset$ then let b' be the point of $I \cap B$ which satisfies $(\langle b_i, b' \rangle \backslash \{b'\}) \cap B = \emptyset$. Otherwise, set $b' = f^{kn}(a_i)$. Since f is A-monotone and $\mathrm{Int}(\langle a_i, b' \rangle) \cap B = \emptyset$ then $f^{kn}|_{\langle b_i, b' \rangle}$ is monotone. Hence $f^{kn}|_{\langle a_i, b' \rangle}$ is also monotone. From property (6) it follows that $f^{kn}(b_i) \in C_a$ and thus $f^{kn}(b') \in C_a$. Suppose that $f^{kn}(b') = a_i$. If $b' \in B$ then $a_i \in A$ — a contradiction. If $b' \notin B$ then $b' = f^{kn}(a_i)$, and thus $f^{2kn}|_{\langle a_i, x \rangle}$ is monotone. So x is monotone equivalent to $a_i \in V(T)$, which contradicts the hypothesis that x is significant. Thus we conclude that $f^{kn}(b') \neq a_i$. Then we define $a_{i+1} \in (\langle a_i, f^{kn}(b') \rangle \backslash \{a_i\}) \cap (V(T) \cup B)$ so that $\langle a_i, a_{i+1} \rangle \cap (V(T) \cup B) = \{a_i, a_{i+1}\}$. Such a point exists by Proposition 4.2. Thus $f^{kn}|_{\langle a_{i+1}, b_i \rangle}$ is a monotone interval map because $a_i \notin B$.

So we have defined the points a_{i+1} and b_{i+1} in each case. These points satisfy properties (2)–(5) by construction, and the sequences are finite because $A \cup V(T)$ is finite. ■

LEMMA 7.2. *Let* $(\mathcal{T}, \Theta)$ *be a pattern, and let* $f: T \to T$ *be a tree map which exhibits the pattern over* A *and is* A*-monotone. Then for each significant periodic point* x *with period* n *and for eac* $k \geqslant 1$ *there exists a unique loop* α *of length* kn *of the* $(\mathcal{T}, \Theta)$*-path graph such that* x *and* α *are associated.*

Proof. By Lemma 7.1, there exists a unique basic path π such that $\langle \pi \rangle$ contains an interval J satisfying $x \in J$, for each $i \leqslant kn$, $f^i(J)$ is contained in the convex hull of a basic path, and $f^{kn}(J) = \langle \pi \rangle$. Lemma 7.1 says that any loop of length kn in the $(\mathcal{T}, \Theta)$-path graph associated to x must start at the basic path π. By Lemma 4.5, it follows that there exists a unique basic path π_1 such that $f(J) \subset \langle \pi_1 \rangle \subset \langle f(\pi) \rangle$. By repeating this argument, we obatin a unique loop of length kn associated to x. ■

From Lemma 7.2 it follows that any loop of length kn associated to x is repetitive and is a concatenation of the unique loop of length n. Given a loop α of length n, we are interested in the existence of a periodic point x associated to this loop. Lemma 7.3 says that this point x is unique (up to monotone equivalence), but we remark that this does not mean that α and z are associated for each point z which is f-monotone equivalent to x.

LEMMA 7.3. *Let* $(\mathcal{T}, \Theta)$ *be a pattern and let* $f: T \to T$ *be a tree map which exhibits the pattern over* A *and is* A*-monotone. Then for each loop* α *of length* n *of the* $(\mathcal{T}, \Theta)$*-path graph there exists a unique* $x \in T$ *(up to monotone equivalence) such that* $f^n(x) = x$ *and* α *are associated.*

Proof. Let

$$\alpha = \pi_0 \to \pi_1 \to \cdots \to \pi_{n-1} \to \pi_0$$

be the loop of length n of the $(\mathcal{T}, \Theta)$-path graph. Since f is A-monotone, for each $i, f(\langle \pi_i \rangle)$ is an interval. Hence there exists a unique subinterval $J \subset \langle \pi_0 \rangle$ such that $f^i(J) \subset \langle \pi_i \rangle$ for $i = 0, \ldots, n-1$ and $f^n(J) = \langle \pi_0 \rangle$. Therefore there exists $x \in J$, unique up to f-monotone equivalence, such that $f^n(x) = x$, and α are associated. ∎

Proof of Theorem C. We start by proving (a). Let x be a significant periodic point of period n. From Lemma 7.2 there exists a unique loop α of length n associated to x. We need to show that this loop is non-repetitive. Suppose that there exists a loop β of length l, where $n = kl$ for some $k > 1$, such that $\alpha = \beta^k$. By Lemma 7.3, there exists z associated to β such that $f^l(z) = z$. In particular, the period of z is strictly less than n. But z is also associated to α, so by Lemma 7.3, it follows that x and z are f-monotone equivalent, which contradicts the minimality of the period of x.

Now we prove (b). Let

$$\alpha = \pi_0 \to \pi_1 \to \cdots \to \pi_{n-1} \to \pi_0$$

be a non-repetitive loop of length n of the $(\mathcal{T}, \Theta)$-path graph. By the statement and the proof of Lemma 7.3, there exists a unique subinterval $J \subset \langle \pi_0 \rangle$ such that $f^i(J) \subset \langle \pi_i \rangle$ for $i = 0, \ldots, n-1$ and $f^n(J) = \langle \pi_0 \rangle$. Furthermore, there exists $x \in J$, unique up to f-monotone equivalence, such that $f^n(x) = x$, and x and α are associated.

Suppose that there is no element of $V(T) \cup A$ which is f-monotone equivalent to x. We will show that there exists a significant point f-monotone equivalent to x of period n associated to α. With the above hypothesis, we start by proving that z and α are associated for each point $z \in T$ which is f-monotone equivalent to x. First we prove that $x \in J$. Assume that $z \notin J$. Then $\langle f^n(z), f^n(x) \rangle$ contains an endpoint of $\langle \pi_0 \rangle$, that is, an element of A. Since A is a finite set and $f^n(z)$ and $f^n(x) = x$ belong to the same f-monotone equivalence class, there exists a periodic point of A in $\langle f^n(z), f^n(x) \rangle$. So x is f-monotone equivalent to an element of A; a contradiction. Since $f^n(z)$ and $f^n(x) = x$ are f-monotone equivalent, by the same arguments we again obtain that $f^n(z) \in J$. By repeating this process, it follows that z and α are associated.

Let y be a periodic point f-monotone equivalent to x with minimal period among all the points in the class of x. Thus y is significant. We have to prove that the period k of y is n. Suppose that $k < n$. Then by part (a) there exists a unique non-repetitive loop associated to y whose length divides k. Then α would be repetitive by Lemma 7.2, which is a contradiction. ∎

To prove Theorem D, we need the following lemma as a technical tool.

LEMMA 7.4. *Let $(\mathcal{T}, \Theta)$ be a pattern, and let $f: T \to T$ be a tree map which exhibits $(\mathcal{T}, \Theta)$. Let*

$$\alpha = \pi_0 \to \pi_1 \to \cdots \to \pi_{n-1} \to \pi_0$$

be a loop of length n of the $(\mathcal{T}, \Theta)$-path graph. Then there exists a finite union $J = \bigcup_{i=1}^{m} \langle a_i, b_i \rangle \subset T$ of intervals whose interiors are pairwise disjoint which satisfies the following properties:

(1) *$x \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_m < b_m \leqslant y$, where $\{x, y\} = \pi_0$ is a basic path and $<$ denotes an orientation of $\langle \pi_0 \rangle$.*

(2) $f^j(J) \subset \langle \pi_j \rangle$ *for* $j = 1, \dots, n-1$ *and* $f^n(J) = \langle \pi_0 \rangle$.
(3) $f^n|_{\{a_1, b_1, \dots, a_m, b_m\}}$ *is monotone.*
(4) $f^n(b_i) = f^n(a_{i+1})$ *for* $i = 1, \dots, m-1$.
(5) $f^n(\langle a_i, b_i \rangle) \subset \langle f^n(a_i), f^n(b_i) \rangle$ *for* $i = 1, \dots, m$.

Proof. This follows immediately from Lemma 3.2 and its proof taking $s = 1$ and $\pi_n^j = \pi_0$. ∎

Proof of Theorem D. Let $J = \bigcup_{i=1}^m \langle a_i, b_i \rangle$ be the finite union of intervals defined in the previous lemma. We shall distinguish two cases.

If $f^n|_{\{a_1, b_1, \dots, a_m, b_m\}}$ is monotone increasing then $f^n|_J : J \to \langle \pi_0 \rangle$ has a fixed point x by standard arguments. Furthermore x and α are associated. Otherwise, $f^n|_{\{a_1, b_1, \dots, a_m, b_m\}}$ is monotone decreasing. By replacing n by $2n$ and α by α^2 there exists a finite union of intervals $J' = \bigcup_{i=1}^k \langle c_i, d_i \rangle$ satisfying the properties of Lemma 7.4, and such that $f^{2n}|_{\{c_1, d_1, \dots, c_k, d_k\}}$ is monotone increasing. As above, $f^{2n}|_{J'}$ has a fixed point x such that x and α^2 are associated. ∎

8. CHARACTERIZATION OF ZERO ENTROPY PATTERNS

In this section we prove Theorem E and Corollary F. Theorem E will follow from the following two propositions.

Proposition 8.1. *Let* $(\mathscr{T}', \Theta')$ *be a reduced pattern of the pattern* $(\mathscr{T}, \Theta)$. *Then* $h(\mathscr{T}, \Theta) = h(\mathscr{T}', \Theta')$.

Proposition 8.2. *Let* $(\mathscr{T}, \Theta)$ *be a non-trivial pattern which is not reducible. Then* $h(\mathscr{T}, \Theta) > 0$.

Proof of Theorem E. Let $(\mathscr{T}, \Theta)$ be a strongly reducible pattern. By Proposition 8.1, we get that $h(\mathscr{T}, \Theta)$ is the entropy of a trivial pattern which is zero. Conversely, if $(\mathscr{T}, \Theta)$ it is not strongly reducible then after finitely many reductions we obtain a non-reducible (non-trivial) pattern. From Propositions 8.1 and 8.2, we obtain that $h(\mathscr{T}, \Theta) > 0$. This ends the proof of the theorem. ∎

Now we prove Propositions 8.1 and 8.2.

Proof of Proposition 8.1. Let $(\mathscr{T}', \Theta') = ([T', A'], [\theta'])$ be a reduced pattern of $(\mathscr{T}, \Theta) = ([T, A], [\theta])$. With the notation from the definition of a reduction, for each $i \in \{1, 2, \dots, k\}$, there exists j_i such that $f(C_i) \subset C_{j_i}$. So the map $f' : T' \to T'$ such that $f' \circ \phi = \phi \circ f$ is well defined. Clearly, f' exhibits the pattern $(\mathscr{T}', \Theta')$ over A'.

We claim that f' is A'-monotone. To prove the claim note that the map $\phi|_{\langle a, b \rangle}$ is monotone for each $a, b \in T$. On the other hand, for each basic path π' of (T, A') there exists a basic path π'' of (T, A) such that $\phi(\langle \pi'' \rangle_T) = \langle \pi' \rangle_{T'}$. Since f is A-monotone, $f|_{\langle \pi'' \rangle}$ is monotone. From the definition of f', we see that $f'|_{\langle \pi' \rangle} = \phi|_{f(\langle \pi'' \rangle)} \circ f|_{\langle \pi'' \rangle}$. So $f'|_{\langle \pi' \rangle}$ is monotone and $f'(\langle \pi' \rangle) = \langle f'(\pi') \rangle$. This ends the proof of the claim.

The next step in the proof will be to show that $h(f|_C) = 0$. By hypothesis, there exists a basic path π of (T, A) such that $(\mathscr{T}, \Theta)$ is π-reducible. So there exists $n \in \mathbb{N}$ such that $f^{n+m}(\pi) = f^n(\pi)$ for some $m \in \mathbb{N}$. Moreover, by the A-monotonicity of f, the map $f^m : \langle f^n(\pi) \rangle \to \langle f^n(\pi) \rangle$ is monotone. Therefore $h(f^m|_{\langle f^n(\pi) \rangle}) = 0$. Now set

$\tilde{C} = \bigcup_{k=0}^{m-1} \langle f^{n+k}(\pi)\rangle$. We observe that $\tilde{C} = \bigcup_{k=0}^{m-1} f^k(\langle f^n(\pi)\rangle)$ since f is A-monotone. By standard arguments (see for instance Lemma 4.1.10 of [1]), $h(f|_{\tilde{C}}) = 0$. Clearly, $f(\tilde{C}) = \tilde{C} \subset C$. Again by the A-monotonicity of f, we have that $C = \bigcup_{k\geq 0} f^k(\langle\pi\rangle)$. Therefore $\tilde{C} = \bigcap_{k\geq 0} f^k(C)$, and hence $h(f|_C) = h(f|_{\tilde{C}}) = 0$ (see for instance Corollary 4.1.8 of [1]).

To complete the proof of the proposition, let M be the maximal closed f-invariant set contained in $T\backslash C$. Since $h(f|_C) = 0$, we obtain that $h(f) = h(f|_M)$. Since ϕ is a homeomorphism from $T\backslash C$ into $T'\backslash\phi(C)$ and $\phi(C)$ is finite, we see that $h(f|_M) = h(f'|_{\phi(M)}) = h(f')$. Furthermore, f' is A'-monotone, and so by Theorem A, we have that

$$h(\mathscr{T}, \Theta) = h(f) = h(f|_M) = h(f') = h(\mathscr{T}', \Theta').$$

This ends the proof of the proposition. ■

Proof of Proposition 8.2. Let $(\mathscr{T}, \Theta) = ([T, A], [\theta])$. By assumption, $(\mathscr{T}, \Theta)$ is not reducible which implies that for each basic path π,

(1) $\langle\theta(\pi)\rangle$ contains at least one basic path,
(2) there exists $n_\pi \in \mathbb{N}$ such that $\theta^{n_\pi}(\pi)$ is not contained in a discrete component. In other words, $\langle\theta^{n_\pi}(\pi)\rangle$ contains at least two basic paths.

Let M denote the path transition matrix of $(\mathscr{T}, \Theta)$. For each $n \in \mathbb{N}$ we will denote the i, j entry of the matrix M^n by $m_{i,j}^n$. From (1) we get that

$$\sum_j m_{i,j}^n \leq \sum_j m_{i,j}^{n+1}$$

for each n and i and from (2),

$$\sum_j m_{i,j}^{n_\pi} \geq 2.$$

Set $N = \max\{n_\pi \colon \pi \text{ is a basic path of } (T, A)\}$. Therefore

$$\sum_j m_{i,j}^N \geq 2$$

for each i. If M^N is irreducible (see [17, Chapter 13] for a definition) then by [17, eq. (37) of Chapter 13] we see that $\rho(M^N) \geq 2$. By Theorem A, this completes the proof in this case. Now assume that M^N is reducible. By [17, Chapter 13, Section 4], M^N can be written as a block matrix of the form

$$\begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix}$$

where M_1 is irreducible. By the above argument, $\rho(M^N) \geq \rho(M_1) \geq 2$, which ends the proof of the proposition. ■

Proof of Corollary F. Since $h([T, A], [\theta]) = 0$, it follows from Theorem E that $([T, A], [\theta])$ is reducible. That is, there exists a basic path π such that $\theta^n(\pi)$ is contained in a single discrete component for each $n \geq 0$. Write $\text{Orb}(\pi) = \{\theta^n(\pi) \colon n \geq 0\}$.

Now we claim that either $\pi_1 \cap \pi_2 = \emptyset$ for each two different basic paths $\pi_1, \pi_2 \in \text{Orb}(\pi)$ or $s = |A|$. Assume that there exists $\pi_1, \pi_2 \in \text{Orb}(\pi)$ such that $\pi_1 \neq \pi_2$ and $\pi_1 \cap \pi_2 \neq \emptyset$. Since A is a periodic orbit (θ is one-to-one) and $\theta(\pi_i)$ is a basic path for $i \in \{1, 2\}$, it follows that $\theta(\pi_1)$ and $\theta(\pi_2)$ are two different basic paths in $\text{Orb}(\pi)$ such that $\theta(\pi_1) \cap \theta(\pi_2) \neq \emptyset$. For each $a \in \text{En}(\langle A\rangle_T)$, there exists an integer l such that $\theta^l(\pi_1) \cap \theta^l(\pi_2) = \{a\}$. Since $\theta^l(\pi_i)$ is contained in a single discrete component for each $i \in \{1, 2\}$ and $\theta^l(\pi_1) \neq \theta^l(\pi_2)$, this endpoint a must be the unique point of A in the corresponding edge of T. Thus the claim follows.

Since A is a periodic orbit, for all $a \in A$ there is a basic path π_a such that $a \in \pi_a$ and $\pi_a \in \mathrm{Orb}(\pi)$. Since $\langle A \rangle_T$ is an s-star with $1 < s \leqslant 3$, from the claim we get that either there does not exist a discrete component containing two different basic paths $\pi_1, \pi_2 \in \mathrm{Orb}(\pi)$ and $|A|$ is even, or $s = |A|$ and A has a unique point in each edge of $\langle A \rangle_T$. In the first case, there is a reduction to a pattern $([T', A'], [\theta'])$, where $|A'| = |A|/2$ and T' is an s'-star with $1 < s' \leqslant 3$. In the second case, there is a single reduction of $([T, A], [\theta])$ to a trivial pattern. Since $([T, A], [\theta])$ is strongly reducible by Theorem E, the corollary follows by repeating this process. ■

Acknowledgements—The authors thank the Institut Non-Linéaire de Nice for its kind hospitality during the visits that originated this paper. They have been supported by the Integrated Action HF-269-B. Lluís Alsedà, Francesc Mañosas and Pere Mumbrú have been partially supported by DGICYT grant number PB93-0860. John Guaschi and Jérôme Los would like to thank the Centre de Recerca Matemàtica, Bellaterra, Spain for its hospitality during the writing of much of this paper. John Guaschi also thanks the Generalitat de Catalunya for its financial support during this period.

REFERENCES

1. L. Alsedà, J. Llibre and M. Misiurewicz: *Combinatorial dynamics and entropy in dimension one*, Advanced Series on Nonlinear Dynamics **5**, World Scientific, Singapore (1993).
2. L. Alsedà and X. Ye: No division and the set of periods for tree maps, *Ergod Th. Dynam. Sys.* **15** (1995), 221–237.
3. D. Asimov and J. Franks: *Unremovable closed orbits*, Geometric dynamics, Rio de Janeiro (1981), 22–29. (This is a revised version of a paper which appeared as SLNM **1007**, Springer, Berlin, 1983.).
4. S. Baldwin: Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions, *Discrete Math.* **67** (1987), 111–127.
5. M. Bestvina and M. Handel: Train tracks for surface homeomorphisms, *Topology* **34** (1995), 109–140.
6. L. Block: Simple periodic orbits of mappings of the interval, *Trans. Amer. Math. Soc.* **254** (1979), 391–398.
7. L. Block, A. Blokh and E. Coven: Zero entropy permutations, *Chaos* **5** (1995), 1331–1337.
8. L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young: *Periodic points and topological entropy of one dimensional maps*, *Global theory of dynamical systems*, pp. 18–34, SLNM **819**, Springer, Berlin (1980).
9. A. M. Blokh: Periods implying almost all periods, trees with snowflakes, and zero entropy maps, preprint, SUNY Stony Brook (1991).
10. R. Bowen: *Entropy and the fundamental group*, SLNM **668**, Springer, Berlin (1978), pp. 21–29.
11. R. Bowen and J. Franks: The periodic points of maps of the disk and the interval, *Topology* **15** (1976), 337–342.
12. P. Boyland: Braid types and a topological method of proving positive entropy, preprint (1984).
13. M. Denker, C. Grillenberger and K. Sigmund: *Ergodic theory on compact spaces*, SLNM **527**, Springer, Berlin (1976).
14. W. Dicks: *Groups, trees and projective modules*, SLNM **790**, Springer, Berlin (1980).
15. A. Fathi, F. Laudenbach and V. Poénaru: Travaux de Thurston sur les surfaces, *Astérisque* **66–67** (1979).
16. J.-M. Gambaudo, S. van Strien and C. Tresser: The periodic orbit structure of orientation preserving diffeomorphisms on D^2 with topological entropy zero, *Ann. Inst. H. Poincaré* (Phys. théor.) **49** (1989), 335–356.
17. F. R. Gantmacher: *The theory of matrices*, Vol. 2, Chelsea, New York (1959).
18. J. Guaschi, J. Llibre and R. S. MacKay: A classification of braid types for periodic orbits of diffeomorphisms of surfaces of genus one with topological entropy zero, *Pub. Mat. UAB* **35** (1991), 543–558.
19. T. Hall: Unremovable periodic orbits of homeomorphisms, *Math. Proc. Cambridge Phil. Soc.* **110** (1991), 523–531.
20. B. Jiang: Nielsen fixed point theory, *Contemporary Math.* **14** (1983).
21. S. Li and X. Ye: Topological entropy for finite inverval sets of Y, preprint, Trieste (1993).
22. J. Libre and R. S. MacKay: A classification of braid types for diffeomorphisms of surfaces of genus zero with topological entropy zero, *J. London Math. Soc.* **42** (1990), 562–576.
23. J. Llibre and M. Misiurewicz: Horseshoes, entropy and periods for graph maps, *Topology* **52** (1993), 149–164.
24. J. Los: Pseudo-Anosov maps and invariant train tracks in the disc: a finite algorithm, *Proc. London Math. Soc.* **66** (1993), 400–430.

25. T. Matsuoka: The number and linking of periodic solutions of periodic systems, *Invent. Math.* **20** (1983), 319–340.
26. M. Misiurewicz: Horseshoes for continuous mappings of an interval, in *Dynamical systems*, Liguori, Ed. Napoli (1980), pp. 127–135.
27. M. Misiurewica and Z. Nitecki: Combinatorial patterns for maps of the interval, *Mem. Amer. Math. Soc.* **94** (1991).
28. A. N. Sharkovskii: Co-existence of the cycles of a continuous mapping of the line into itself (in Russian), *Ukrain, Math. Zh.* **16** (1) (1964), 61–71.
29. J. R. Stallings: Topology of finite graphs, *Inv. Math.* **71** (1983), 551–565.
30. W. P. Thurston: On the geometry and dynamics of surfaces, *Bull. Amer. Math. Soc.* **19** (1990), 417–431.
31. L.-S. Young: On the prevalence of horseshoes, *Trans. Amer. Math. Soc.* **263** (1981), 75–88.

Department de Matemàtiques
Universitat Autònoma de Barcelona
Bellaterra, 08193 Barcelona, Spain

Laboratoire de Mathématiques Emile Picard
U.M.R. CNRS 5580
Université Toulouse III
118 route de Narbonne
31062 Toulouse Cedex, France

I.N.L.N., URM CNRS 129
Université de Nice-Sophia Antipolis
1361 Route des Lucioles
Sophia Antipolis, 06560 Valbonne, France

Department de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585, 08071 Barcelona, Spain